Exploring the Efficacy of a Spatial Reasoning Approach to Teaching and Learning Fractions in the Early Years

A thesis submitted in fulfilment of the requirements for the degree of

Doctor of Philosophy

Chelsea Cutting

B.Ed Deakin University

M.Ed University of South Australia

School of Education

College of Design and Social Context

RMIT University

Australia

July 2023

Declaration

I certify that except where due acknowledgement has been made, this research is that of the author alone; the content of this research submission is the result of work which has been carried out since the official commencement date of the approved research program; any editorial work, paid or unpaid, carried out by a third party is acknowledged; and ethics procedures and guidelines have been followed.

In addition, I certify that this submission contains no material previously submitted for award of any qualification at any other university or institution, unless approved for a joint-award with another institution, and acknowledge that no part of this work will, in the future, be used in a submission in my name, for any other qualification in any university or other tertiary institution without the prior approval of the University, and where applicable, any partner institution responsible for the joint-award of this degree. I acknowledge that copyright of any published works contained within this thesis resides with the copyright holder(s) of those works.

I acknowledge the support I have received for my research through the provision of an Australian Government Research Training Program Scholarship.

Chelsea Cutting

18 July 2023

This thesis is dedicated to my sons, Ollie and Beau. Being your mum will forever be my greatest achievement. I love you both beyond the universe.

Acknowledgements

I acknowledge Capstone Editing and Johanna Bowman who provided copyediting and proofreading services, according to the guidelines laid out in the university-endorsed national ‘Guidelines for Editing Research Theses’.

I am incredibly grateful for the opportunity to complete such inspiring work. My sincere thanks go to the teachers and children that participated in this research. Without their support and commitment, this study would not have been possible. This project would also not have been possible without my supervisors, Emerita Professor Dianne Siemon and Dr Angela Rogers, who supported and believed passionately in this work from the very first day of my candidature. I am grateful for your wisdom, knowledge, and unwavering belief in me throughout this project.

I would like to thank Professor Tom Lowrie for his mentorship and support over the past several years. I am so grateful for the opportunities I’ve had to work with you and your research team, which has developed my thinking and understanding beyond measure. In addition, I am incredibly thankful for the friendship you have all extended to me.

I extend my sincere appreciation to Associate Professor Kevin Larkin, who has provided invaluable advice on all aspects of my study, read many drafts of my work, and provided much wisdom (and humour) throughout this journey. In addition, Associate Professor Jodie Miller has been a great inspiration and support throughout this project, particularly on the juggle between academia and family life. I am so grateful for the support and friendship you have both provided me.

I would like to thank my colleague, Dr John Kennedy for the quantitative analysis support provided for my project, in addition to my Mathematics Education colleagues, particularly

Professor Amie Albrecht, Associate Professor Simon Leonard, Dr Lisa O'Keeffe, Dr Florence Gabriel, and Dr Sam Fowler for their support and encouragement. I would also like to thank Professor Barbara Comber, Associate Professor Victoria Whitington, and Professor Shane Dawson for their support during my candidature and lecturing position at UniSA. My sincere thanks extend to my Mount Gambier UniSA colleagues who have cheered me on all the way to the finish line, while providing comic relief, coffee, and wine when it was required. We are a small, but tight-knit group and I am so grateful I get to work with you all every day.

I cannot thank my friends and family enough for their support and encouragement throughout this period. To my sister Teneille, your support and encouragement has been unwavering. I could not have done this without you. To my parents, and parent-in-laws; each have supported me in many ways, not least of which included looking after the boys when needed.

Finally, but most importantly, I thank my incredible husband and our beautiful boys. While we have faced many challenges over the past few years, you have always been my source of strength and motivation. **Thank you,** for your endless love and support.

Publications from this Research Project

Cutting, C. (2019). Re-thinking fraction instruction in primary school: the case for an alternative approach in the early years. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics Education Research: Impacting Practice Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 204-211). MERGA.

Cutting, C. (2021). Spatial reasoning and the development of early fraction understanding. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways Proceedings of the 43rd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 171-178). MERGA.

Table of Contents

List of Figures

List of Tables

List of Abbreviations

ACARA	Australian Curriculum Assessment and Reporting Authority
ACfM	Assessment for Common Misunderstandings
CESA	Catholic Education South Australia
CLASS	Components of the Classroom Assessment Scoring System
DBR	design-based research
ITND	integrated theory of numerical development
NAPLAN	National Assessment Program Literacy and Numeracy
NRC	National Research Council
PISA	Programme for International Student Assessment
PST	pre-service teacher
RPS	ratio processing system
TBI	task-based interview
TIMMS	Trends in International Mathematics and Science Study
WNB	whole number bias

Abstract

Fractions are widely recognised as one of the most difficult areas of the school mathematics curriculum to teach and learn. A deep understanding of fractions supports further learning in areas such as algebra, proportional reasoning, and statistics. In Australia, research has established that many students in the primary and middle years of schooling experience considerable difficulty in recognising, naming, and renaming common fractions. This difficulty is consistently reflected in Australian students' results on international assessments of mathematical literacy.

One issue contributing to these difficulties, is the way in which fractions are represented in the Australian Mathematics Curriculum content descriptors. Within the early years, children are expected to recognise equal parts of a whole, primarily emphasising the part-whole meaning of fractions. This focus promotes an additive, counting-based approach to the teaching and learning of fractions. Such an approach overlooks the critical foundations fractions have in fair sharing contexts and masks the multiplicative nature of fractions.

The foundation of early fraction understanding needs to be based on fair sharing or equal partitioning. Fair sharing is an intuitive idea that very young children experience in a range of contexts before they begin school. Research on children's early number learning suggests that fair sharing is a highly spatial activity. This is particularly evident when children share collections of items, or partition continuous objects fairly based on attributes such as geometric similarities and spatial arrangements.

Spatial reasoning and its association with mathematics achievement is well established in the literature. Although there is increasing interest in examining young children's spatial

reasoning and mathematical development in general, there is very little research that has examined the role spatial reasoning may play in relation to early fraction learning. This thesis aims to address this gap in the research.

A Design-Based Research (DBR) methodology was employed, to explore the extent to which young children could demonstrate an understanding of a broader range of fraction meanings, experienced through a spatial reasoning approach. Consistent with DBR, a local instruction theory was developed to frame the design and implementation of an iterative teaching experiment that introduced children to the *fraction as an operator, fraction as a measure* and *fraction as a relation* meanings. The local instruction theory was represented by a series of conjectured key indicators that proposed a pathway for developing the three meanings of fractions through a spatial reasoning approach. The key indicators informed the design of an intervention program comprising of 13 consecutive 60-minute lessons, which were implemented with 70, Year 1 and 2 children at three primary schools in regional South Australia. In its final form, the four key indicators of the local instruction theory were: *creating and justifying equal shares*; *reinitialising the unit*; *recognising proportional equivalence* and *connecting multiplicative relations*.

The results of the study revealed that young children developed rich understandings of the three meanings of fractions, which were directly supported by spatial reasoning. The children demonstrated an understanding of the *fraction as an operator* meaning, through a focus on spatial visualisation to predict and justify the outcome of sharing geometric shapes, and small collections of objects. The children developed the *fraction as a measure* meaning specifically in the way they were able to work with unit fractions, composite units, and their ability to name different fraction parts. *Spatial structuring* was a critical construct that supported the children's ability to make connections about the associated *fraction as a measure* ideas, and fraction magnitude.

Further, the children demonstrated an ability to recognise fraction and proportional equivalence, through contexts that developed their spatial proportional reasoning. The children were able to work with early *fraction as a relation* ideas, specifically simple ratio, which was supported by spatial structuring. Additionally, the influence of spatial structuring within *fraction as a relation* contexts, positively influenced children's part-part relations for whole number quantities.

Another critical finding of this study was the way in which children used gesture to communicate both spatial and mathematical knowledge. The children's spontaneous use of gesture throughout the intervention provided greater insights into their use of spatial reasoning strategies, and how these were connected to their understanding of fractions. This is an important finding that makes a significant contribution to what is currently known about how young children use self-initiated gestures to explain and communicate their mathematical thinking.

As the three meanings of fractions are not typically taught or considered to be entirely appropriate for children in the early years, this study challenges the foundations of the current curriculum expectations that promote additive approaches to learning fractions. The findings provide powerful insights into what is possible in the early years of schooling. Specifically, the ability for children to develop a multiplicative foundation for an extended range of fraction meanings, though a spatial reasoning approach. Given the persistent difficulties children in the middle and upper years of primary education experience with fractions, an explicit emphasis on spatial reasoning in the early years of school provides an important alternative approach to the teaching and learning of factions.

Chapter 1: Introduction

As a primary school teacher, a parent and now an educational researcher, I have always been fascinated with how children learn and experience mathematics. This fascination stems from my own educational experiences and reflections on how I made sense of mathematics growing up. This introductory chapter will outline the journey I took throughout my educational experiences that provided the motivation for this PhD research project.

1.1 Personal Experiences

Growing up, I did not engage with how mathematics was taught at school. I was taught using a procedural-based approach where the focus was on memorising facts and arithmetic procedures. This meant I found maths lessons quite disconnected from real-life experiences. I never excelled in the subject, and I certainly never felt confident in my abilities throughout my schooling. However, I was very musical from an early age, learning and playing several instruments largely by ear with high levels of success. It was not until my undergraduate studies that I recognised that the reason I was good at music was because I could 'see' the structure of the compositions I was playing and had memorised. At its heart, music involves understanding fractions, whole numbers, and proportional relationships. I discovered that I had developed these mental models of music, which now I realise is the ability to visualise complex mathematical relationships. Yet, I could not understand why my teachers throughout primary and secondary schooling did not capitalise on this understanding to promote my mathematical abilities. During my undergraduate studies, I recognised these connections and, subsequently, these experiences paved the way to developing a love and passion for providing children with meaningful and rich mathematics education.

Throughout the early stages of my classroom teaching career (2007–2010), I had the opportunity to be involved in a sustained Numeracy Project as part of a Catholic Education South Australia (CESA) initiative. The project was designed to support teachers in exploring evidenced-based pedagogical approaches to teaching and learning mathematics through an action research project.

While the project enabled me to examine most aspects of my mathematics teaching program, I gained several insights into children's fraction understanding that eventually led to this PhD study. During my time on the Numeracy Project, I noticed that young children would engage in fair sharing contexts in an informal, yet relatively accurate manner. I observed that during play, they could create a balanced mixture of sand and water, estimating and adjusting the proportions to ensure the mix was neither too wet nor too dry for building a sandcastle. Another observation was that young children could share items among several people fairly, and visually estimate and divide lumps of playdough into relatively equal parts when playing—even redistributing parts if more or fewer shares were required. When, however, similar tasks were set in classroom contexts, children failed to consistently demonstrate this ease in creating fair shares and distribution. This was highlighted to me in an activity that required a length of rope to be partitioned into quarters and then eighths by pegging fraction symbols onto the rope. The activity was introduced to my Year 3 class at the time (~8–9 years old), as follows:

> This rope represents a pathway between the front of the school building and the start of the carpark. Our principal wants four light posts to be installed along the path equally to make it safer at night. Can you use pegs and post-it notes to label where each light will go? What do you notice about the position of each light? Can you describe how the pathway was divided?

After the children had completed this component of the task, I posed the following extension:

> Uh oh! The groundsman accidentally ordered eight lights, not four! Our principal said he would like us to install all eight lights to make the path as bright as possible. If we had to divide this pathway equally to for all eight lights, how can we use the previous task to help us? What do you notice about the number of lights and the distance between them?

Many of the children at the time stated that there was ‘not enough rope’ to partition into eighths after they had quartered the rope. This difficulty indicated that the children considered eighths to be the same size as quarters when stating that they could not ‘fit’ eighths along the length of rope. This suggested a lack of experience with making and naming fractions and the ability to visualise the relationship between the number of parts created from a given whole and their size.

In another teaching experience with a group of children exploring odd and even numbers, a common definition children provided was ‘even numbers can be divided into two equal groups, odd numbers cannot’. In response to this statement, during one such lesson, I asked one child the following question:

> What if you and I had three chocolate bars to share between us? As there are three in total, does that mean we cannot share them fairly between the two of us? Would you just let me have two, and you have one?

In this instance, the child could determine that this was not a fair outcome. Quickly, the child suggested we cut the third bar in half, so we each received 1-and-a-half chocolate bars, thus indicating they could easily visualise this fair sharing context. However, when I asked the child to think back to their definition of odd and even numbers, the child was perplexed, suggesting they had little experience articulating the relationship between fractions as numbers more generally. Although the children in these examples had been exposed to fraction ideas in the previous two years of schooling (such as partitioning continuous wholes and discrete sets into halves, quarters, and eighths), their thinking suggested a lack of understanding between partitioning in a range of different fractional contexts and representations. These difficulties surprised me, as I had witnessed the children's ability to engage in many of these ideas with ease during their play, so I expected they would have been able to clearly articulate an appreciation of the outcome of sharing situations and the size of parts created in relation to one another.

After completing the Numeracy Project, I started my Master of Education by coursework. There was a small research component of the degree where I investigated children's development of time concepts. I chose this topic to research because I was interested in how children visualise and develop their understanding of an area of mathematics that they cannot physically see, hear, touch, or manipulate—they can only experience. In this small study, I found that using number lines and multi-link cubes to represent the relationships between time units (such as minutes, hours, quarter, and half hours) greatly enhanced children's ability to visualise the mathematical ideas of time when learning to read analogue clocks, which also included an appreciation of fractions.

During my time as a classroom teacher, I also started lecturing at the University of South Australia in the Mathematics Education courses for the Early Childhood and Primary initial teacher education programs. I was encouraged to apply because of my involvement in the CESA Numeracy Project and the completion of my Master of Education. What I soon discovered was that overwhelmingly, the pre-service teachers (PSTs) demonstrated a lack of confidence in their understanding of fractions, referring to them as confusing and difficult. What appeared to be a common factor in PSTs' lack of confidence and competence was their inability to visualise and reasonably estimate the size of different fractions. For example, when PSTs were asked to explain, without calculating, which fraction was bigger (e.g., $\frac{6}{7}$ or $\frac{7}{8}$ presented as symbols), or if asked to predict a reasonable outcome for dividing and multiplying fractions, the vast majority stated the only way they could do it was to remember the procedures and formulas required for each problem (i.e., finding the common denominator, or invert and multiply respectively). Few could estimate or provide a justification for their answer. In many ways, the PSTs exhibited the same problems I saw with the children in the early years of primary school; that is, there seemed to be difficulty in visualising the size of the fractions they were dealing with and predicting the outcomes of dividing different quantities more generally. Clearly, the development of fractions had been a difficult process throughout their primary and secondary schooling experiences, which provided another reason for me to examine this issue.

1.2 The Issue

The difficulties children (and PSTs) experience with fractions are not isolated to my own teaching experiences. A 2013 report by the Queensland Studies Authority revealed that many students in Years 7 and 9 were demonstrating difficulty with early fundamental concepts and ideas for fractions (such as partitioning and the part-whole relationship), which are expected to be mastered in the early years of schooling (Queensland Studies Authority, [QSA], 2013). In 2015, 70% of Australian Year 4 children achieved the ‘intermediate’ international benchmark for the Trends in Mathematics and Science Study (TIMMS) for mathematics—the second-lowest achievement band of the test. In TIMMS, the children demonstrated the weakest understanding in the domain of number, compared to geometry, measurement, and data domains (Thomson et al., 2017b). Within the number domain, TIMMS assessed the topics of whole numbers, fractions, decimals and expressions, simple equations, and relationships. In the most recent TIMMS assessment (2019), 22 countries outperformed Australian Year 4 students in mathematics.

Concerningly, Year 4 TIMMS mathematics results in Australia have not improved since 2007 (Thomson et al., 2020).

Similar trends are reported in the Programme for International Student Assessment (PISA) data. For example, results suggested that students in Australia have been declining in their mathematical capabilities since 2003 (Thomson et al., 2017a). Similarly, the National Assessment Program Literacy and Numeracy (NAPLAN) data suggest that for the Year 3 Numeracy assessment, there has been no evidence of national improvement in the scores over the past decade (McGaw et al., 2020).

In a large-scale project commissioned by the Victorian Department of Education and Training, in partnership with Catholic Education Commission of Victoria, in early 2000, the following difficulties in fractions exhibited by many children in Years 5 to 9 were identified:

- ‘Reading, renaming, ordering, interpreting, and applying common fractions, particularly those greater than 1.
- Reading, renaming, ordering, interpreting, and applying decimal fractions.
- Recognising the applicability of ratio and proportion and justifying this mathematically in terms of fractions, percentages, or written ratios.’ (Siemon et al., 2000, p. 21).

Siemon (2003) argues that partitioning is the missing link between young children’s experiences with fractions in the early years of schooling and their performance and capabilities for multiplicative and proportional reasoning in later years. Partitioning is the ‘behaviours that create equal-sized groups… [in which] division is most directly derived from equi-partitioning, with multiplication following as its inverse, rather than the traditional view that multiplication precedes division’ (Confrey et al., 2009, p. 347). While Siemon’s (2003) research reports on middle school data from nearly 20 years ago, the current national and international mathematics assessment data suggests these issues are still plaguing primary and middle school children today. This implies that partitioning and fair sharing are not being adequately established in the early years of schooling. My experiences as a junior primary classroom teacher and a PST lecturer suggest this is still the case.

1.2.1 Fractions: A Critical Mathematical Domain

A strong understanding of fractions is foundational for engaging with a range of other mathematical ideas, such as decimal fractions, percentages, rates, ratios, and proportions. These are also required for more complex domains of mathematics such as algebra, proportional reasoning, and statistics (Hilton et al., 2016; Lamon, 2007; Pearn & Stephens, 2016; Stohl, 2005; Siegler et al., 2011). Fractions, therefore, act as a gateway to more complex areas of mathematics and without a sound understanding, children will struggle to engage in these areas of mathematics in upper primary and secondary school (Hilton et al., 2016). However, the importance of understanding and working flexibly with fractions is not just confined to the classroom. An understanding of fractions is required for engagement in a myriad of contexts in our daily lives. For example, understanding interest rates on loans and investments, calculating different measures such as time and distance (and speed which is the associated ratio of these two measures); through to cooking a meal which often requires an understanding of the different quantities of ingredients required (typically presented as fractions and decimal fractions). Yet this area of mathematics, although embedded early in school curricula and prevalent in many of our daily activities, presents a significant challenge to many children—and often their teachers (Fuchs et al., 2013; Hansen et al., 2017; Xie & Masingila, 2017).

As noted earlier, I was surprised early in my teaching career by the difficulties children exhibited with fractions, given their importance, and children’s intuition and experiences in their daily play activities. I soon realised that inhibiting factors were contributing to such difficulties, which still exist now. The first was from a curriculum perspective, where the content for fractions in the Australian Curriculum, Mathematics version 8.4 (Australian Curriculum Assessment and Reporting Authority [ACARA], n.d) is quite scarce. For example, there is no content explicitly related to fractions in Foundation (the first year of formal schooling). In Year 1, children were only expected to identify representations of 1-half. However, there is no mention of whether this is in continuous or discrete contexts. By the end of Year 2, children are only required to divide small collections and shapes into halves, quarters, and eighths. Not only is the content limited within these year levels, but the primary emphasis in the expectations is on simple, proper fractions and the part-whole meaning only. What is also evident in the curricula demands is that there is no connection to how fractions and whole number ideas are related or integrated.

Given the PISA, TIMMS and NAPLAN results mentioned above, it would seem that these problems may be due, in part, to the types of early mathematical experiences to which children in Australia are exposed. It appears that, this is impacting on their ability to work with more complex areas of mathematics in the later years of primary school and beyond. Yet children's play would suggest we can capitalise on their informal mathematical experiences and, use these opportunities to develop a range of meaningful fraction ideas and understandings.

1.3 Possibilities

My fascination with young children's mathematical development and the recognition that they can engage in powerful mathematical ideas earlier than expected, led me to examine theories of number development more broadly. Over the past 50 years, there has been extensive research conducted on young children's rational number reasoning from the fields of neuroscience, psychology, and mathematics education. Children's early fraction, whole number, and proportional reasoning capabilities from 'the brain level to the classroom level' (Obersteiner et al., 2019, p. 135) have been explored. Yet, there appears to be a lack of integration across and between these disciplines specifically related to young children's fraction development (Bruce et al., 2017; Obersteiner et al., 2019). This disconnect means that much of the research is confined to discrete fields and does not allow the translation of theory into practice (Bruce et al., 2017).

Adding to this disconnect, there are differing perspectives on how young children understand fractions. For example, Steffe (2001) believes that children's fraction development is built on their whole number knowledge based on a foundation of counting and measurement. However, other researchers such as Pepper (1991) and Confrey et al. (2014b) suggest that young children's fraction understanding develops independently or in parallel to their whole number and counting abilities through an awareness of equality and fair shares. This perspective is supported by research from the neuroscience and psychology fields. For instance, a range of studies conducted with children in the pre- and early years of school identified that young children have an appreciation of non-symbolic and non-numerical fraction quantities and ratios. For example, children can identify proportional relationships in non-symbolic contexts, such as matching water and juice ratios between different size containers or identifying halving and doubling relationships between geometric shapes. These findings suggested that there is a spatial reasoning element to young children's development of quantity implicit in these perspectives. My hypothesis of this connection led me to the large body of work that suggests strong spatial reasoning abilities are predictive of children's success in mathematics and science at school and beyond (Kell et al., 2013; Newcombe, 2010; Uttal & Cohen, 2012). Despite the powerful influence spatial reasoning has on general mathematics performance (Bruce et al., 2015b; Mix & Cheng, 2012; Mulligan, 2015; Sinclair & Bruce, 2014; Wai et al., 2009) and the implied use of spatial reasoning in children's early experiences of fractions, there is a lack of research on how it may help children develop an extended range of fraction understandings. This gap in current research implied to me that there is an opportunity to explore innovative and integrated theoretical interventions exploring the role spatial reasoning can play in young children's fraction development. This context provided the motivation for the present study.

1.4 Organisation of the Thesis

This study explores to what extent and in what ways young children can develop an extended range of fraction meanings through a spatial reasoning approach. A Design-Based Research (DBR) methodology was employed to explore this problem through a series of iterative teaching experiments. Through the construction and refinement of a local instruction theory (Gravemeijer & Van Eerde, 2009), an intervention program taught in several early years' classrooms provided the basis for contributing new theoretical and practical insights for young children's fraction development. The presentation of this study is organised in the following format.

Chapter Two: This chapter provides a critical review of the research literature concerning the teaching and learning of fractions in the early years of primary school. In addition, the review considers the difficulties many children face in developing a conceptual understanding of fractions. A range of theoretical perspectives on the development of fraction understanding are considered and critiqued.

This chapter also examines interdisciplinary literature that discusses spatial reasoning and its role in assisting in developing mathematical knowledge, particularly fractions. In contrast to the literature on children's fraction understanding, which is predominantly derived from the mathematics education domain, the spatial reasoning constructs related to fraction development are generated from a modest number of studies from the fields of psychology and neuroscience.

The discussion and critical analysis of these theoretical frameworks are used to justify the approach that underpins this study.

Chapter Three: This chapter discusses the theoretical paradigm of interpretivism, and the research methods used in this educational study. Design-based research (DBR) is discussed and justified within the broader interpretivist research paradigm that guides the study’s design. Three DBR phases—preparation, teaching experiment and retrospective analysis—are explained. Next, the development of the intervention program based on a conjectured local instruction theory is presented, based on a synthesis of the literature in Chapter 2. The process of how the intervention program was designed, implemented, and refined iteratively throughout the study is explained. Participants, data collection methods and analysis are described. Finally, trustworthiness and ethics considerations are discussed.

Chapter Four: As part of Phase One (the preparation phase), the results from the pilot of the intervention program are described. This chapter discusses how the tasks were trialled for inclusion in the intervention program and insights into how children engaged with the concepts, skills and ideas are analysed. Findings are used to explain and justify the refinement of the local instruction theory and intervention program.

Chapter Five and Chapter Six: The analyses and findings from Phase Two (the teaching experiment) are presented in respective chapters. The findings illustrate how the children in each Class A and Class B engaged with the intervention program and how this provided evidence to support the local instruction theory. Minor refinements were made during each iteration (characteristic of undertaking DBR), including a change in timing and delivery of the intervention program for Class C, due to the COVID-19 pandemic unfolding during this time.

Chapter Seven: This chapter moves the thesis into Phase Three (the retrospective analysis) where the study’s results are discussed, and the local instruction theory confirmed as a consequence of this study. The research questions are answered and a discussion of the findings in relation to the literature is undertaken to interpret the insights identified. The significance of this study for teaching fractions in the early years of primary school is detailed.

Chapter Eight: The study's implications are discussed from pedagogical and curricula perspectives. The chapter details the limitations of the study, including that more broadscale research is required to strengthen the theoretical and practical outputs empirically. Recommendations on how this study's design could be enhanced for greater fidelity and the organisational mechanisms required to better support teachers in developing fraction knowledge in the future are presented. The thesis concludes with recommendations for future research and my final reflections.

Chapter 2: The Development of Early Fraction Understanding

2.1 Chapter Overview

This study explores the theoretical perspectives on how children develop a range of fraction ideas and meanings in the early years of primary school to consider the role spatial reasoning may play in this development. As described in Chapter 1, there are concerns over children's mathematical competencies in Australia, including their fraction understanding and ability to work with an extended range of fraction ideas throughout primary and secondary school (Callingham & Siemon, 2021; Siemon, 2016; Thomson et al., 2020). Recent national and international assessment data suggests that children are not developing the fundamental ideas required to work with an extended range of rational number concepts, which requires understanding the relationships between whole numbers and fractions. Moreover, as introduced in the previous chapter, the requirements of the Australian Curriculum (version 8.4) suggest that children are not being provided with sufficient opportunities to explore and develop early fraction ideas because of the limited content evident in the early years of the curriculum.

Research into the nature and acquisition of rational number knowledge and, more specifically, fractions, is extensive, yet the problems children typically experience with fractions remain consistent in the research concerning this important area of mathematics, especially in Australia (Thomson, 2020). This chapter reviews research on how children come to understand fractions and the implications this has for teaching and learning rational number in the early years.

To explore this educational issue, the chapter begins by defining what fractions are through discussing the underpinning concepts and the multiple meanings of fractions in section

2.2. This section explores how children typically develop these concepts and fraction meanings and what conceptual and pedagogical issues are raised in the literature concerning young children’s experiences. Section 2.3 identifies and critically discusses contemporary theories of early rational number development and how these theories influence different approaches to fraction instruction. Based on the analysis of these theories, common characteristics that underpin young children’s engagement with early fraction ideas are identified.

Advances in neuroscience and psychology suggest that young children engage with such ideas in non-symbolic, spatial contexts; therefore, spatial reasoning and the relationship to fraction development are explored in section 2.4. The role multimodal representations play in developing early fraction knowledge is addressed in section 2.5, as representations play a critical role in mathematical development generally within the early years. However, this section also reflects on the role of representations in relation to the spatial reasoning influences considered in the previous section, to include a discussion on gesture. Section 2.6 presents the central insights gained from this literature review about young children’s potential for learning an extended range of fraction ideas and states the research questions formulated as a result of this review. Section 2.7 concludes the chapter with a summary of the key understandings generated from this review.

2.2 Exploring the Complexities of Fractions

Fractions play a crucial role in mathematical development. Theoretically, they are essential because they require a flexible and deeper understanding of quantity beyond what is typically experienced with whole numbers (Siegler et al., 2011). Educationally, a deep understanding of fractions is imperative because of the fundamental role they play in developing more complex mathematical concepts, such as algebraic thinking and proportional reasoning (Empson et al., 2006; Hackenberg, 2013; Lamon, 2012). Research has established that a solid understanding of fractions at a primary school level has strong predictive outcomes for future mathematical competency (Confrey, 2012; Hilton et al., 2016; Lortie-Forgues et al., 2015; Siegler et al., 2012). Despite the wealth of research conducted in this area (e.g., Behr et al., 1992; Booth & Newton, 2012; Confrey et al., 2014b; Ni & Zhou, 2005; Steffe & Olive, 2010; Siegler et al., 2012), many children and adults experience considerable difficulty with reading, recognising, and renaming fractional quantities in different contexts (Siemon, 2003; Siemon et al., 2006). Current evidence suggests that children are not engaging with fraction ideas that allow them to build sophisticated understandings represented through the various meanings of fractions (Cortina et al., 2014; Thomson et al., 2020). Cortina et al. (2014) state, ‘there remains great dissatisfaction with the typical levels of fraction understanding attained by most students and, more importantly, with what is known about how to improve the situation’ (p. 1). To explore these issues, the underpinning concepts, and meanings of fractions in terms of the difficulties children commonly experience in their development will now be discussed.

2.2.1 Underpinning Concepts for Fraction Understanding

It is widely acknowledged by many authors working in the field of fractions that there are three fundamental concepts for developing fraction understanding: *partitioning*, *unitising,* and *quantitative equivalence* (e.g., Confrey & Maloney, 2010; Kieren, 1993; Lamon, 1996, 2007; Pitkethly & Hunting, 1996). Partitioning is considered the foundational concept for working with fractions because it is underpinned by the process of fair sharing—whether a group of objects or single item—and is an activity that very young children engage with in their typical play experiences (Confrey, 2012; Lamon, 1996). Partitioning is also referred to as *equipartitioning* by some researchers (referring to the necessity for all parts to be equal, e.g., Confrey, 2012). For the purposes of the present study, the term ‘partitioning’ implies equal parts and is used throughout this thesis.

2.2.1.1 Partitioning

The process of dividing a whole or unit into equal-sized parts is described as partitioning (Lamon, 2012; Mack, 2001). However, it requires more than just the act of creating equal parts or groups; it is the ability to recognise the relationship between the number of equal parts and the size and name of the parts created (Siemon, 2003). Additionally, partitioning is the ability to make generalisations about how fractions might be recreated from a single whole and renamed (e.g., increasing the number of parts by a certain factor means the same factor increases the required number of parts; Siemon et al., 2012). On this basis, it is the foundation to the related constructs of division and multiplication, ratio, rate, and proportional reasoning (Confrey et al., 2009; Siemon, 2003).

In the context of early childhood, children typically experience this concept through fair sharing (such as creating equal shares of cakes, pizzas, groups of lollies, marbles, etc.), where they start to develop an understanding of the relationship between the number of shares and the size of the share (Lamon, 1996; Mitchelmore & White, 1995; Siemon, 2013). That is, they start to experience and appreciate the idea that when they increase the number of shares of a collection or object, each share becomes smaller.

Holt et al., (2012) describe three essential criteria that children must coordinate successively in the process of partitioning.

1) Create the correct number of groups or parts,

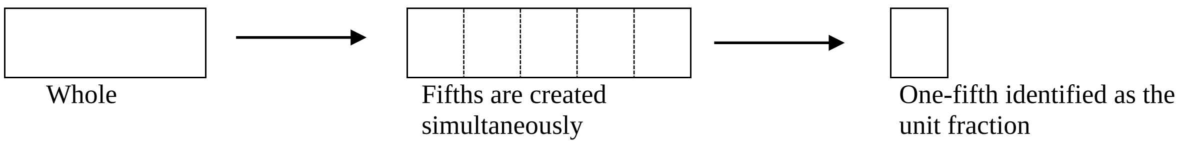
2) Generate equal-size groups or parts, and

3) Exhaust the collection or whole (p. 484).

Multiplicative partitioning, also referred to as splitting by some researchers (e.g., Confrey et al., 2014b), is the ability to see or consider the creation of the parts within a whole. For example, partitioning an area model into fifths involves seeing five equal parts embedded within the whole, and then disembedding the part form the whole in the fair sharing process to identify the unit (Bruce, 2013; Hackenburg & Lee, 2012), as illustrated in Figure 2.1.

Figure 2.1

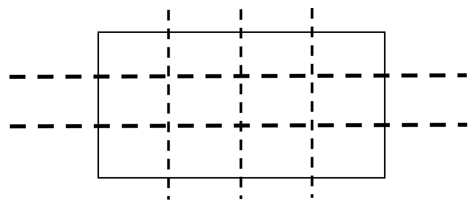
An Interpretation of Multiplicative Partitioning (Adapted from Bruce, 2013)



Several researchers suggest that multiplicative partitioning provides a foundation for the development of children’s understanding because it is derived from the appreciation of division and reassembly of division (i.e., multiplication), as opposed to whole number knowledge and counting (Confrey, 1994/2014b; Empson et al., 2006; Kieren, 1988, 1995; Mack, 2001; Siemon, 2003). For example, the repeated act of halving is considered as a critical entry point for children to explore the multiplicative foundations of partitioning, because it is an intuitive process very young children are capable of (Siemon, 2013; Confrey, 2012). It introduces children into the notion of fraction families, that is, repeated halving creates all fractions in the halving family, (*halves, quarters, eighths,* etc); repeated *thirding* creates all of the fractions in the third family, (*thirds, ninths,* etc). Bruce (2013) describes further that this initial understanding allows for children to explore composite units, such as partitioning by a four split and then a successive three split to create *twelfths*, as illustrated in Figure 2.2. This understanding is critical for understanding the multiplicative foundations of fractions more generally (Bruce, 2013; Siemon, 2013).

Figure 2.2

An Example of Composite Units Created from Partitioning (Adapted from Bruce, 2013)



An examination of teaching approaches in relation to partitioning is discussed later in this thesis.

2.2.1.2 Unitising

Lamon (2006) described unitising as the 'cognitive assignment of a unit of measurement to a given quantity, referring to the "size chunk" one constructs in terms of which to think about a given commodity' (p. 79). That is, it is an understanding that the parts created from the act of partitioning can be named as a quantity or unit, depending on how the whole has been partitioned. This definition suggests that quantity is described in two parts—the number or 'how many' parts are identified and the unit of measure or 'how much' each part is worth. Unitising is essential in the development of place value, multiplicative thinking, and fraction understanding (Steffe & Olive, 2010) because it allows us to consider a whole as a composition of multiple units. For example, a carton of 12 eggs could be thought of as one unit of 12 eggs, two units of six eggs (six eggs being the unit of half the set), four eggs as a unit of one-third of 12 and so on. Confrey and Smith (1995) use the term *reinitialising* to name this concept, where they describe it as a process of working with 'unit of units', such as the egg example.

Several researchers state that young children's early understandings and experiences of fair sharing a whole with a focus on creating different units appears to be an effective way to understand the multiplicative foundations of fractions (Confrey & Smith, 1995; Confrey et al., 2014a; Lamon, 2006; Mack, 2001). Lamon (2006) also suggests that children's flexibility in unitising or reinitialising through exploring partitioning has not been emphasised as a basis for formal fraction instruction, potentially contributing to the difficulty children exhibit when learning this topic.

2.2.1.3 Quantitative Equivalence

Quantitative equivalence necessitates that children recognise that two or more different fraction terms can represent the same quantity, thus belonging to an equivalent set (Wong & Evans, 2007). For example, the quantitative equivalence set of fractions for $\frac{3}{4}$ can be represented symbolically as $\frac{6}{8}, \frac{9}{12}, \ldots \frac{30}{40}$ and so on. Pedersen and Bjerre (2021) propose two conceptions of quantitative equivalence that are required to fully understand this concept: *proportional equivalence* and *unit equivalence*. The first is grounded in proportionally equal relationships between fractions, whereby children explore how different wholes, such as half an apple and half a bag of marbles, represent the same fractional quantity in relation to their respective whole. Unit equivalence is concerned with distinguishing equivalent fractions within the same whole. That means, recognising 2-quarters of an apple is equivalent to 1-half of the same apple, or a ratio of 8 flowers and 4 vases is equivalent to 12 flowers and 6 vases. The distinction between these two forms of equivalence have not necessarily been articulated widely in the present literature; however, they are both important constructs to develop as they lead to conceptual (multiplicative) understandings of rational number more broadly (Brousseau et al., 2004; Lamon, 2012; Pedersen & Bjerre, 2021).

An understanding of quantitative equivalence is necessary for children to develop as it supports an appreciation of the density of fractions, which is an understanding that fractions can be represented infinitely and can have an infinite number of successors (unlike whole numbers).

Moreover, quantitative equivalence is required for operating with fractions, such as finding the lowest common denominator when adding or subtracting; however, it is also required to work flexibly with ratios and proportion.

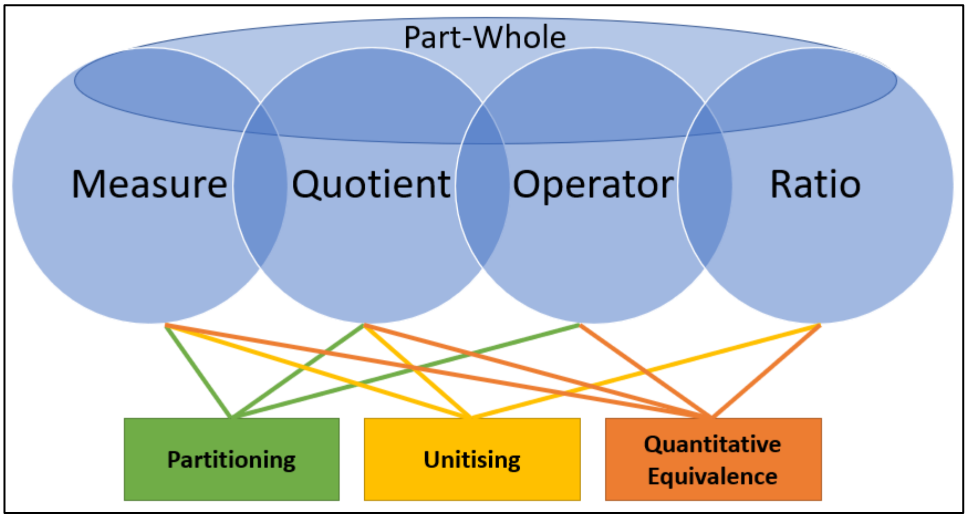
In summary, partitioning, unitising, and quantitative equivalence are the three underpinning concepts required to develop a sound understanding of fractions. However, there are various ways fractions can be used to represent different meanings of quantity, which means that the three concepts interact with each meaning in different ways. The common meanings of fractions and how the concepts are related to each will now be explored, including a discussion on what is known about young children's experiences and capabilities of each meaning.

2.2.2 The Multiple Meanings of Fractions

Fractions are a subset of rational numbers (i.e., of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$). Kieren (1976) was one of the first researchers to articulate the different meanings of rational numbers, identifying that fractions are not a single construct but, rather, a set of interconnected sub-constructs, conveying varying meanings (Charalambous & Pitta-Pantazi, 2007). Kieren (1976) initially proposed seven constructs of fractions, which were intended to describe what rational number conceptual understanding might involve. These initial constructs were identified as *decimal*, *equivalence*, *ratio*, *operator*, *quotient, measure,* and *procedural understanding* (in the form of fraction algorithms). This initial model was later refined to a framework that described five fraction meanings and their relationship to the three underpinning concepts described above. The meanings were identified as *part-whole*, *measure*, *quotient*, *operator,* and *ratio* (Kieren, 1993). Later, the model was further redeveloped so that the part-whole meaning of fractions was an embedded component of each of the four other meanings (*measure*, *quotient*, *operator,* and *ratio*), as illustrated in Figure 2.3.

Figure 2.3

An Interpretation of Kieren's (1993) Fraction Framework



The connection between each of the fraction meanings and the three concepts of partitioning, unitising, and quantitative equivalence are indicated by the corresponding line segments. For example, the measurement and quotient meanings are underpinned by all three concepts. The ratio meaning is developed with the concepts of quantitative equivalence and unitising, while an understanding of the operator meaning requires conceptual knowledge of partitioning and quantitative equivalence. The rationale of the part-whole interpretation being an influential component of the four remaining meanings of fractions is said to form the basis for identifying the relationship of all parts to their whole(s) in the context of each meaning (Charalambous & Pitta-Pantazi, 2007).

An important component to developing the different meanings of fractions is that they should be explored and understood in both continuous and discrete contexts (Confrey, 2012; Lamon, 1999). That is, fraction models can be used to describe numbers that indicate a count or mass, such as a group of children (discrete) or a container of water (continuous; Bloom & Wynn, 1997; Rapp et al., 2015). The same model can, however, be used in both contexts, such as distinguishing 3-quarters of a plate of sandwiches as a discrete collection of sandwiches, as opposed to 3-quarters of a single sandwich as a continuous model. Understanding and working with both discrete and continuous models across all fraction meanings is an important landmark in early fraction understanding because these ideas require different cognitive demands (Confrey & Maloney, 2010). The demands include recognising that a continuous model is the formation of multiple, contiguous parts, and the discrete model involves the need to visualise finite parts within one set.

The identified meanings of fractions in this model are considered to have ‘stood the test of time’ in the explication of rational number meanings (Pitkethly & Hunting, 1996), specifically the importance of the concepts of partitioning, unitising, and quantitative equivalence which have been supported extensively in other works (e.g., Behr et al., 1981; Behr et al., 1984; Confrey & Maloney, 2010; Hunting, 1983; Lamon, 2012; Pothier & Sawada, 1990; Siemon, 2003). Each of the fraction meanings and relevant literature exploring conceptual and pedagogical perspectives in the teaching and learning of these meanings will now be discussed.

2.2.2.1 Fraction as Part-Whole

Fraction as part-whole is defined by the following essential components: the whole, which is decomposed into equal parts identifying a unit (such as $\frac{1}{4}$), and the complementary part that, when reconstructed with a unit, completes the whole (i.e., $\frac{3}{4}$). The part-whole meaning of fractions is a useful introduction to the idea that a whole quantity is partitioned into equal-sized parts such as partitioning a sandwich in half (continuous whole) or the class into quarters (discrete whole) and that a part has a direct relationship to the whole and its complementary part (Baturo & Cooper, 1999). This meaning it is one of the most common used in introducing and teaching fractions because children’s prior experiences of fractions are derived from fair sharing, which is the creation of parts from a whole (Confrey et al., 2014b; Getenet & Callingham, 2017; Pitkethly & Hunting, 1996; Middleton et al., 2015; Siemon et al., 2015).

2.2.2.2 Conceptual and Pedagogical Issues

While important to building fraction knowledge across all meanings of fractions, Lamon (2001) describes that focusing primarily on the part-whole meaning is undesirable, as it provides a narrow pathway to understanding of the structure and breadth of rational numbers. There seems to be an implicit assumption that this is the easiest and most important meaning for learning fractions because of its prevalence in teachers' preferred methods of teaching this area of mathematics (Gould, 2013; Stevens et al., 2020; Wilkie & Roche, 2022; Zhang et al., 2013). However, according to Mamede et al. (2005), there is no supporting evidence for this assumption, and it is strongly refuted by some researchers (see Confrey et al., 2014b; Nunes & Bryant, 2007; Lamon, 2007). In their large-scale work, Behr et al. (1983) declare that the part-whole meaning of rational numbers, along with the process of partitioning, to be fundamental in developing an understanding of all fraction meanings. Other researchers (see Confrey et al., 2014a; Mack, 2001; Streefland, 1993b) reiterate that teaching practices must transcend this introductory meaning of fractions to develop a true understanding of the multiplicative nature of fractions. Despite the wealth of research undertaken in the development and construction of fractions, it seems this message has been lost as teaching practices continue to rely almost exclusively on the use of part-whole situations (Behr et al., 1992; Clarke et al., 2006; Gould, 2005; Webster, 2020). This reliance on the part-whole meaning is said to contribute to 'difficulties in working with fractions operations and even algebraic reasoning' (Norton & Hackenberg, 2010, p. 343).

The reason for the part-whole meaning being described as the least valuable meaning, is primarily due to how this meaning is often explored. For example, a common approach to teaching the part-whole idea emphasises taking a 'double count' approach (Gould, 2011). That is, too often, children are taught to count the number of shaded parts (usually of a regular 2D shape) and record as the numerator. They are then instructed to count the total number of parts and record the count as the denominator. This approach treats each component of the fraction as a cardinal quantity rather than viewing the symbol as a representation of the relational components of a quantity. While the part-whole meaning requires parts to be partitioned equally, equality refers to the development of the quantitative equivalence concept. Therefore, a part may consist of more than one piece of the whole (Lamon, 2012) such as 2-quarters are equal to 1-half, one and a half is equal to 3-halves and so forth. It is these cognitively demanding ideas supporting an understanding of fractions as part-whole, which should be the focus rather than the double count strategy too often embedded into early instruction (Clarke et al., 2008; Lamon, 2012; Reeder & Utley, 2017; Siebert & Gaskin, 2006).

This difficulty relating to the way the part-whole meaning is often introduced contributes to what is described as whole number bias (WNB; Braithwaite & Siegler, 2018; Ni & Zhou, 2005), which is the tendency to focus on or consider the numerators and denominators as independent, whole number quantities. WNB is often evident in such contexts where children might state $\frac{2}{5}$ is less than $\frac{3}{9}$ because the individual digits are 'smaller' as cardinal numbers in the first fraction, which seems to be derived from their part-whole experiences that promote a counting approach. What the literature suggest is that the part-whole meaning is not an appropriate focus or at least starting point for young children in their initial development of fractions, despite its prevalence in current curricula foci and subsequent pedagogical approaches to teaching in the early years of primary schooling.

2.2.2.3 Fraction as Measure

The measurement meaning of fractions considers fractions as numbers, indicating countable, multiplicative units of measurement; therefore, this meaning focuses on developing an appreciation of quantitative magnitude. It requires different thinking to the part-whole meaning because it explores the idea that an infinite number of smaller units can be created by subdividing the current units used to measure a nominated quantity, thus, evoking a ‘multiplicative comparison in which the measuring unit is imagined apart from the thing measured’ (Thompson & Saldanha, 2003, p. 16).

The measurement meaning requires two interrelated ideas. The first is that fractions are considered a number and therefore convey the quantitative magnitude of how big the fractional number is in addition to a measure assigned to an interval (Charalambous & Pitta-Pantazi, 2007; Thompson & Saldanha, 2003). For example, the measurement meaning not only requires an understanding of unit fraction iteration (i.e., $\frac{1}{n}$ as a quantity measure or amount that is iterated *m* times) but also an understanding of $\frac{m}{n}$ as a number or quantity in its own right, where improper fractions (i.e., $m > n$) are explored, such as *10-thirds* is the same as *3 and 1-third* (Simon et al., 2018; Tzur, 1999). Thompson and Saldanha (2003) explain that coming to terms with these two ideas demonstrates a conceptual leap in measurement understanding, which is the ‘realisation that the magnitude of a quantity (its “amount”) as determined in relation to a unit does not change even with a substitution of unit’ (p. 17). In other words, changing the unit of measure of a pizza, for example, to 2-halves rather than 4-quarters does not change the pizza’s absolute magnitude.

The measure meaning is underpinned by all three concepts of partitioning, unitising, and quantitative equivalence as the notion of infinite quantities existing between any two numbers requires the ability to partition and unitise such quantities, recognising the relationships and equivalent measures that exist between them (Hannula, 2003; Lamon, 1999). The fraction as a measure meaning is considered necessary to achieve proficiency in the operations of fractions and for thinking about fractions as a size or magnitude relation (Brousseau et al., 2004; Charalambous & Pitta-Pantazi, 2007; Lamon, 1999; Hackenberg et al., 2016; Stafylidou & Vosniadou, 2004).

2.2.2.4 Conceptual and Pedagogical Issues

The complexity of the fraction as a measure meaning is said to be underestimated by many teachers (Baturo & Cooper, 1999; Lamon, 2007; Payne, 1976; Wilkins & Norton, 2018), who may be unaware of the different types of misconceptions children possess when working with this meaning. Common difficulties, as identified by Mitchell and Horne (2011), include limited part-whole knowledge in the context of fraction as a measure (i.e., the inability to recognise fractional numbers *between* whole numbers such as three and a half is between three and four); incorrectly calculating a decimal fraction, for example, assuming that $1\,\frac{2}{3} = 1.23$ (indicative of the WNB); and counting the zero point on a number line as a unit fraction (e.g., when counting by fourths, referring to zero as the first count of 1-quarter). While it is important to conceptualise a fraction as a measure (i.e., as a quantifiable unit), counting iterations is essentially additive (Gould, 2011), which is not necessarily recognised when exploring these ideas and perpetuates the double count problem discussed in the previous section.

Further, the multiplicative aspects of this meaning can be often reduced to a repeated addition context, which hides the change of unit, treating it as additive counts of iterating a unit, as described above (Davydov & Tsvetkovich, 1991). As Bobos-Kristof (2015) explains, repeated addition is misleading because it reduces multiplicative relationship in which quantity is represented. An example of this is where children may focus on *how many* unit fractions or parts there are (such as 3-quarters is three ‘parts’) rather than thinking about *how much* 3-quarters represents as a quantity in relation to the whole.

There is a close connection between the part-whole and measure meaning, where the part and the referent unit are separate entities. The fraction quantity or measure is constructed from the relationship between the part to be measured and the identified whole (Wantanabe, 2002). Conversely, the part-whole interpretation has the parts embedded within the whole; therefore, when using area or number line models for fraction as a measure, a clear understanding of the referent unit is required (Ball, 1993; Kieren, 1980). However, Clarke (2011) found limited time is spent in Australian schools exploring the measurement interpretation (specifical contexts of $\frac{m}{n}$ where $m > n$), likely due to perpetuating the part-whole and/or the additive limitations described above.

2.2.2.5 Fraction as Quotient

The fraction as quotient meaning considers a quantity as $a \div b$ (Behr et al., 1983). Building on the early sharing experiences likely experienced early in a child’s life, the fraction as quotient meaning results in equal shares of a quantity representing both the division operation and the amount each person receives (Mamede & Oliveira, 2011). For example, in sharing three chocolate bars between four people, $\frac{3}{4}$ is interpreted as the quotient of three divided by four and the resulting fair share. The size of the fraction in this meaning is limitless, as ‘the numerator can be smaller, equal to or bigger than the denominator, and subsequently, the quantity that results from the fair-sharing activity can be less than, equal to or more than the unit’ (Charalambous & Pitta-Pantazi, 2006, p. 299). This meaning requires an understanding of the partitioning and unitising concepts as the parts obtained by the fair sharing activity need to be identified and established as well as the numerical outcome obtained as a result (Kieren, 1999). That is, children need to develop the understand that sharing eight cookies between four people (partitioning) results in 8-quarters ($\frac{8}{4}$) which signifies unitising, and that this share is quantitatively equivalent to or a or two cookies per person.

2.2.2.6 Conceptual and Pedagogical Issues

Children intuitively explore fair sharing situations early on in their mathematical development; however, fraction as quotient is said to be the forgotten meaning of fraction instruction (Clarke, 2006). The underrepresentation of the meaning is not due to the lack of research on fractions as a quotient, particularly for older primary school children (see Empson, 2001; Gould et al., 2006; Siemon, 2003); rather, the limited impact research regarding fraction as quotient has had on classroom practice (Clarke, 2006). However, ‘sharing’ as an idea of division and whole number partitioning has been part of the Australian Curriculum: Mathematics (Australian Curriculum Assessment and Reporting Authority [ACARA] 2018), in Year 1 since 2010 indicating perhaps that teachers are also not recognising how this idea connects with this fraction meaning. Further, given that vast majority of research into this meaning of fraction involves children in the upper primary years of school and beyond, this suggests that teachers do not consider it appropriate or relevant for young children.

Including fraction as quotient ideas early in children’s schooling experiences is supported by Mamede et al. (2005), who explored 6 – 7-year-old children’s equivalence and ordering ideas within fraction problems that were either part-whole or quotient focused, exemplified in Table 2.1.

Table 2.1

Part-Whole and Quotient Problem Pair for Equivalence Example

Conceptual focus	Fraction meaning	Problem
Equivalence	Part-Whole	Bill and Ann each have a bar of chocolate of the same size; Bill breaks his bar into two equal parts and eats 1 of them; Ann breaks hers into four equal parts and eats 2 of them. Does Bill eat more, the same, or less than Ann?
	Quotient	Group A, formed by two children, have to share 1 bar of chocolate fairly; group B, comprising of 4 children, have to share two bars of chocolate fairly. Do the children in group A eat the same, more, or less than the children in group B?

Mamede et al. (2005) found that children had greater success in the fraction as quotient context than the fraction as part-whole, suggesting that early partitioning experiences (i.e., fair sharing) supported their understanding in this meaning. The way the part-whole problem is written also suggests that children may rely on a counting strategy to solve as discussed above, which would likely result in an incorrect response. An additional study by Mamede (2008) also revealed that during a teaching experiment with 6 – 7-year-old children, those who were exposed to part-whole problems were only able to label simple, common fractions, while those who were exposed to partitioned quantities in fraction as quotient contexts were able to order, identify equivalence and correctly label a variety of common fractions. For example, they were able to recognise that $\frac{1}{2}$ was the quotient of sharing one chocolate bar between two people, therefore recognising that $\frac{1}{3}$ means the parts are smaller because the 'one bar' is being shared between more (three) people. Similarly, the children were able to recognise that two children sharing one chocolate bar was equivalent to four children sharing two chocolate bars. Moreover, the findings from this study suggested the children's ability to order fractions successfully in the fraction as quotient problems were based on their ability to unitise the quantities and determine equivalence between these shares. These examples demonstrate that the fraction as a quotient meaning is appropriate and accessible by young children. However, the fraction as quotient meaning is only implicit in the early years of the Australian Curriculum (version 8.4), in the form of sharing a collection of objects in two equal groups for Year 1 and to recognise that a group of objects can be shared into various equal groups for Year 2.

Mamede et al.'s (2005) findings support the idea that children develop informal knowledge of the logic of division from everyday life without instruction in school. This suggests knowledge and awareness of fraction as quotient and the ideas it represents in terms of sharing and connecting to how the quantity is represented and named need to be better developed from the beginning of primary school.

2.2.2.7 Fraction as Operator

Fraction as an operator can be thought of as a function that transforms another quantity, such as identifying $\frac{3}{4}$ of the class of 20 children, or increasing the quantity of a recipe by half, as examples. Lamon (1999) describes several common contexts in which the fraction as an operator meaning is explored: (a) transformations that lengthen or shorten line segments (b) the increase or decrease the number of elements in discrete sets and (c) the proportional transformation (i.e., scaling) of geometric figures. For example, a geometric figure could be enlarged $\frac{p}{q}$ times, or as you could have $\frac{p}{q}$ times-as-many objects (Behr et al., 1983; Confrey, 2008). This type of thinking requires an understanding and application of multiplication and division ideas. For example, to find $\frac{3}{4}$ of 12, the following operations are possible: divide 12 by four to obtain the unit fraction of the set then multiply that quantity by three to determine the quantity of the three units (quarters), or multiply 12 by three, then divide by four.

2.2.2.8 Conceptual and Pedagogical Issues

Several researchers indicate that fraction as an operator is not a common interpretation taught in primary school (Bruce et al., 2014; Usiskin, 2007). Reducing the importance and experience children have of fractions as operator is problematic because it impedes the understanding required for the algebraic application of fractions, which is based on multiplicative relations. Not exploring this meaning presents children with significant and avoidable difficulties as they continue to pursue mathematics throughout their schooling (Bruce et al., 2014). For example, in the context of finding 1-third of six, Hunting et al. (1991) state:

> A fraction such as one-third when viewed as an operator does not destroy the number six and produce two as a result: it operates on six, leaving six unchanged, to produce two, which co-exist with and can be compared with the six from which the two were derived. (p. 77)

However, this type of understanding about the fraction as an operator meaning is often replaced with applying procedural operations on rational numbers (such as multiplying a quantity by the numerator and diving by the denominator). For this reason, it is not explored in the early years which leads to a lack of understanding about how this meaning is interpreted and how rational numbers work more generally (Carraher, 1996; Hunting et al., 1996). As Zhang (2016) suggests, this meaning is more complex than perhaps other fraction meanings because it requires an understanding of part-whole comparisons of the quantity being operated on, a fraction as quotient understanding for how the operation will enlarge or shrink the quantity in question, and fraction as a measure understanding to make sense of the quantity generated in relation to the initial whole.

There is very little literature on young children's capabilities with fraction as an operator problems. However, Empson (1999) found that Year 1 children can successfully operate on a range of quantities, such as calculating 1-quarter of 12 cupcakes or demonstrating an understanding of how to calculate and represent $\frac{2}{3}$ of a packet of 15 pencils. Although this evidence suggests that children in the early years of school are capable of engaging with this meaning of fraction, it does not appear in the Australian Curriculum explicitly until Year 6, again suggesting that there is a disconnection between children's capabilities and their opportunity to develop an extended range of fraction ideas in the early years of primary school.

2.2.2.9 Fraction as Ratio

Fraction as a ratio is the comparison of two quantities (Carraher, 1996), such as number of boys in relation to girls in a class, or the quantity of water in relation to rice required when cooking. The fraction as ratio meaning is often considered in the same way as a *rate* but there is a distinction between the two. Ratio is the comparative index between two of the same quantities (such as a mixture of two liquid measures; two groups of people) and rate is the comparison between two *different* quantities—like time and money as an example (Charalambos & Pitta-Pantazi, 2007; Lamon, 1999). This distinction is important as the ratio meaning of fractions requires an understanding of relative amount or how the invariant property applies. That is, while the quantities may change, the relationship between them remains the same, which connects to the concept of quantitative equivalence. For example, a juice and water mixture may require 1-third of a cup of juice to 2-thirds of a cup of water. To double this mixture means both quantities double (resulting in 2-thirds of a cup of juice to 1-and-a-third cups of water), thus preserving the ratio between the two quantities regardless of the absolute quantity of liquid. On the other hand, rates do not necessarily have an invariant relationship between the two quantities or units of the fraction. An example of this is a monetary investment, where an exponential increase in investment can result from a recurring deposit of the same amount over time due to compound interest.

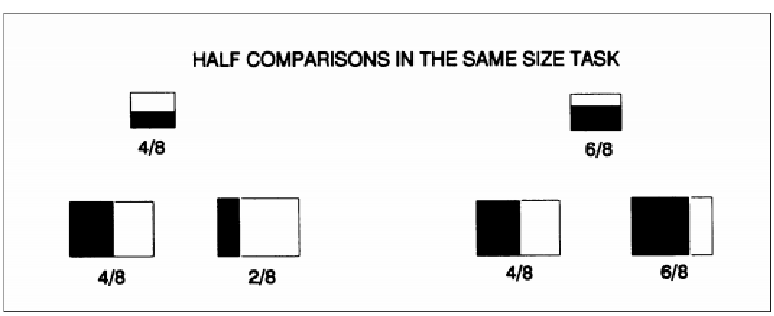
2.2.2.10 Conceptual and Pedagogical Issues

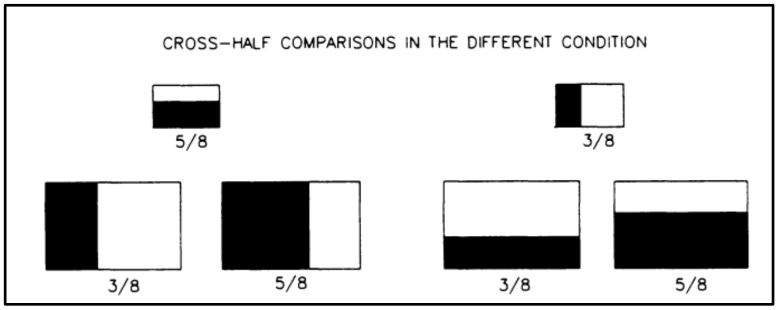
Fraction as a ratio is not explicitly taught until Year 7 in Australia (ACARA, n.d), yet young children encounter this idea early in their play experiences. An example of young children's engagement is in the sandpit, mixing and maintaining a consistent ratio of sand and water for the perfect sandcastle mixture, therefore drawing on the ratio meaning of fractions.

A number of studies support the perspective that young children can engage with fraction as ratio ideas in non-symbolic contexts. For example, Spinillo and Bryant (1991) conducted an experiment where 5 – 7-year-old children were asked to compare a model figure, part of which was black and part of which was white, to two other figures, one of which had the same ratio of black to white colouring (see Figure 2.4). The children were asked to determine which of the two figures had the same ratio of colour, as represented in the model. In the experiment, the fraction symbols seen in Figure 2.4 were not presented in the stimulus; they are presented here for ease of interpretation by the reader.

Figure 2.4

Images Used by Spinillo and Bryant (1991)





Note. From ‘Children’s proportional judgments: The importance of ‘half’, by Spinillo, A. G. & Bryant, P. (1991). *Child Development*, *62*(3), 427–440. Reprinted with permission.

In Spinillo and Bryant’s (1991) study, children aged 6 – 7 years successfully responded beyond chance levels. This revealed the importance of the ‘half’ boundary or proportional benchmark that children relied on in their non-symbolic proportional reasoning, even for multiple halving contexts in the $\frac{6}{8}$ example above, where children’s responses revealed they were visually benchmarking a *half of a half* (1-quarter) to determine the correct proportion. That study concluded that the 6 – 7-year-old children treated these problem contexts as spatial ratios based on their part-part ‘benchmark’ of half in each object. It is important to note that the children in that study had no understanding of symbolic notation relating to fractions, thus these findings suggest a strong connection between the children’s conception of half and their proportional judgement.

Similar results have been obtained by other researchers, exemplifying children’s ability to work successfully work with non-symbolic ratio and proportional contexts (see Goswami, 1989; Huttenlocher et al., 1999; Mix et al., 1999; Singer-Freeman & Goswami, 2001; Schlottmann, 2001). For example, Singer-Freeman and Goswami (2001) found children as young as 4 years of age were able to identify equivalent spatial ratios (i.e., $\frac{2}{4}$ of a pizza versus $\frac{4}{8}$ of a pizza, which refers to equivalence) and although less frequently, these children could also recognise proportional equivalence (i.e., half a box of chocolates is proportionally equivalent to half a pizza). This suggests that young children begin to develop emerging understandings of ratio and proportion long before working with numerical and symbolic representations of ratio (Carraher & Schliemann, 1991). Also revealed by these studies is that the fraction as ratio ideas not only emerge as a perceptual and spatial interpretation of the context but that this developing knowledge is not dependant on children's counting, symbolic understanding, or arithmetic knowledge (Carraher & Schliemann, 1991; Hunting et al., 1996; Pepper, 1991). The issue regarding perceptual and spatial interpretations in the development of fraction as ratio will be discussed later in this chapter; however, evidence suggests that this type of fraction meaning is accessed through young children's spatial reasoning abilities.

2.2.3 Summary of Research on Fraction Meanings

This section has provided insights into how the three concepts of partitioning, unitising and quantitative equivalence are connected to each of the fraction meanings. While Kieren (1988, 1993) and the researchers of the Rational Number Project (Behr et al., 1992, 1993; Lesh et al., 1987) have provided extensive insights into the part-whole, measure, quotient, operator, and ratio meanings of fractions in middle and upper primary schooling, other researchers have provided evidence that suggest young children can engage successfully with all of the meanings.

Specifically, the literature suggests that young children are capable of engaging in each of the meanings of fractions but exhibit difficulties that appear to be perpetuated by over exposure to the part-whole meaning of fractions. The studies that explored young children's capabilities demonstrated that the children could engage in non-symbolic, spatial contexts to explore and represent the various fraction meanings. However, there is limited literature exploring young children's understandings of some of the various meanings of fractions (such as operator and ratio), and prior studies are primarily experimental rather than interventionist in design, meaning what is known about children's potential for developing such ideas is unclear.

To explore how these studies may inform an intervention research design, an examination of theoretical perspectives of how rational number knowledge develops more broadly will now be presented. This discussion examines literature from different disciplines, such as mathematics education, psychology, and neuroscience, to understand from a range of viewpoints how and when young children typically develop fraction ideas and how appropriate and realistic it is to expose young children to an extended range of fraction meanings.

2.3 Theoretical Perspectives on Rational Number Development

In response to the literature examined above on the complexities of each of the meanings of fractions and what is currently known about young children's engagement with each, a cross-disciplinary examination of theories of rational number development is discussed in this section to interpret more broadly the development of fraction knowledge. Vamvakoussi et al., (2018) describe the need for dialogue between the various discipline areas in the following statement:

> Numerical cognition is a research area that appeals to mathematics education researchers, to cognitive-developmental psychologists and to neuroscientists. However, the researchers coming from these different fields approach numerical cognition in different ways in terms of theoretical perspectives, questions asked, methodologies used, and most importantly, of end goals (Berch, 2016). Thus, there is great need for dialogue between psychological and educational research, particularly when it comes to implications for instruction. (p. 84)

Children's fraction understanding is one the most researched area of mathematics, yet the difficulties children experience across all age groups with this topic appear to be largely unimproved over this time. This section explores the foundations of early rational number development through several theoretical perspectives across the various disciplines to determine the implications for instruction relevant to this study.

2.3.1 An Integrated Theory of Number Development

Siegler et al. (2011) proposed an integrated theory of numerical development (ITND), whereby children's numerical development is based on understandings of real numbers, 'including learning the functions that connect the increasingly broad and varied set of numbers to their magnitudes' (p. 274). The ITND asserts that learning mathematics entails the gradual broadening of the set of numbers that can be represented—both whole and fractional—within a single framework (Siegler et al., 2013). From this perspective, learning fractions requires the simultaneous reconceptualisation of quantity and number (Geary, 2006; Leslie et al., 2008; Siegler et al., 2013). This theory is founded on four successive trends, that develop from infancy until adulthood:

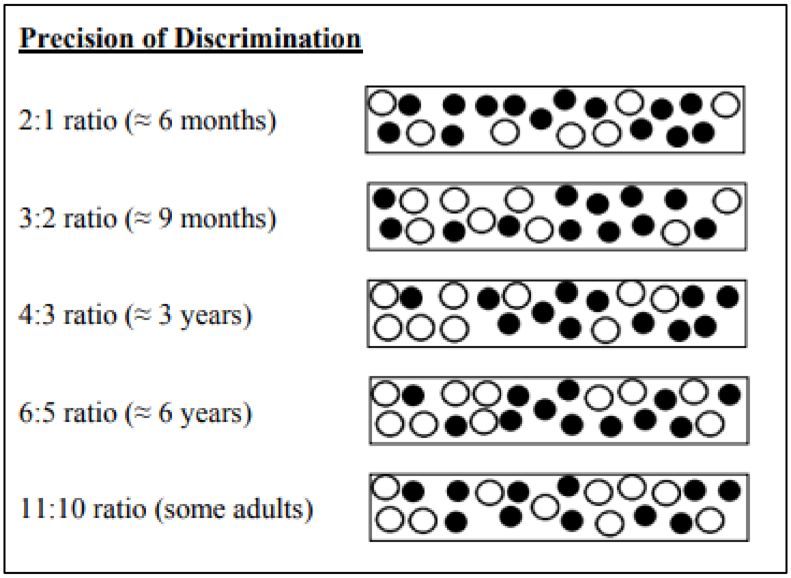
- representing non-symbolic numerical magnitudes increasingly precisely
- linking non-symbolic and symbolic representations of small whole numbers
- extending the range of numbers whose magnitudes are accurately represented to larger whole numbers
- representing accurately the magnitudes of rational numbers, including fractions, decimals, percentages, and negatives (Siegler & Lortie-Forgues, 2014, p. 13).

2.3.1.1 Representing Non-Symbolic Numerical Magnitudes Increasingly Precisely

Siegler and Lortie-Forgues (2014) describe that children as young as 6 months of age possess an awareness and sensitivity to non-symbolic ratios. Figure 2.5 presents the approximate development of this aspect of the ITND. The figure describes the typical age ranges that children develop a sensitivity to the following ratios, presented as dot figures.

Figure 2.5

Proposed Development of Non-Symbolic Numerical Magnitudes from Infancy to Adulthood



Note. From 'An integrative theory of numerical development,' by Siegler, R. S., & Lortie-Forgues, H. (2014), *Child Development Perspectives 8*(3), 144-150. Copyright Robert Siegler and Hugues Lortie-Forgues. Reprinted with permission.

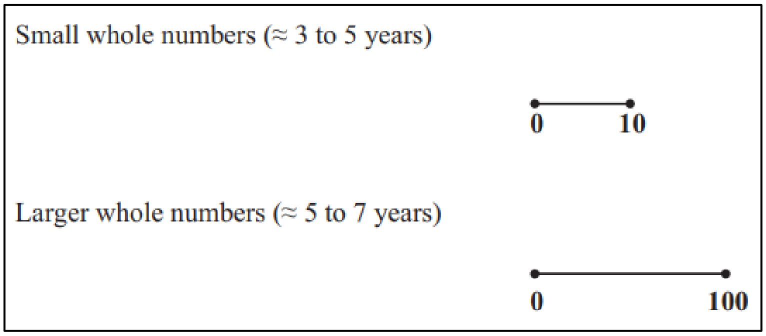
While the ability to discriminate non-symbolic numerical magnitudes, such as those in Figure 2.5 may appear to be a trivial or irrelevant skill in terms of children's global cognitive and physical development, the literature suggests that individual differences in this ability at 6 months are related to mathematical achievement at 3 years of age, even after statistically controlling for IQ (Mazzocco et al., 2011; Siegler & Lortie-Forgues, 2014; Starr & Brannon, 2015).

2.3.1.2 Linking Non-Symbolic and Symbolic Representations of Small Whole Numbers

Children from the ages of 3 – 5 years will also start to represent and identify small whole number magnitudes to 10 through perceptual awareness—such as subitising—and then, from approximately 5 – 7 years, extend this understanding of representing and ordering whole numbers symbolically to 100, as represented in Figure 2.6.

Figure 2.6

Children's Proposed Conceptions of Numbers to 100 Between Three and Five Years of Age



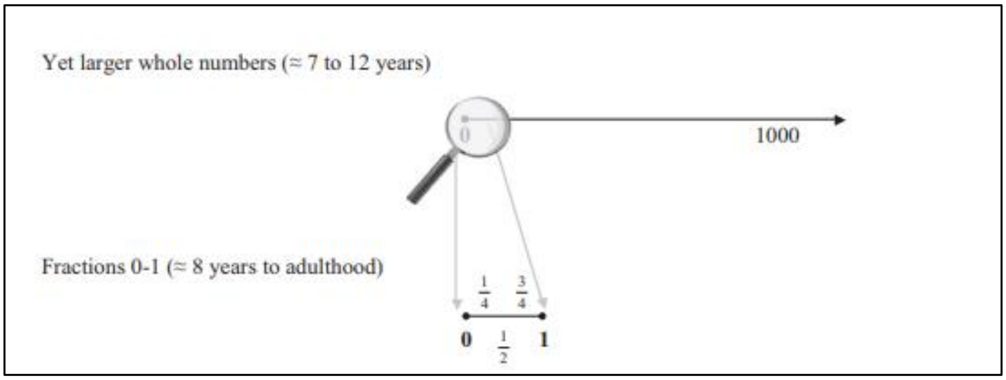
Note. From 'An integrative theory of numerical development,' by Siegler, R. S., & Lortie-Forgues, H. (2014), *Child Development Perspectives 8*(3), 144-150. Copyright Robert Siegler and Hugues Lortie-Forgues. Reprinted with permission.

2.3.1.3 Extending the Range of Numbers Whose Magnitudes are Accurately Represented to Larger Whole Numbers

As children acquire the understanding of the order and magnitude of whole numbers to 100 by the age of approximately 7 years of age, the ITND suggests that children continue to extend their whole number understandings to larger numbers to 1,000 and beyond, while at the same time developing an appreciation for quantities smaller than one, as illustrated in Figure 2.7.

Figure 2.7

Children's Proposed Conceptions of Whole Number and Fractions to Adulthood

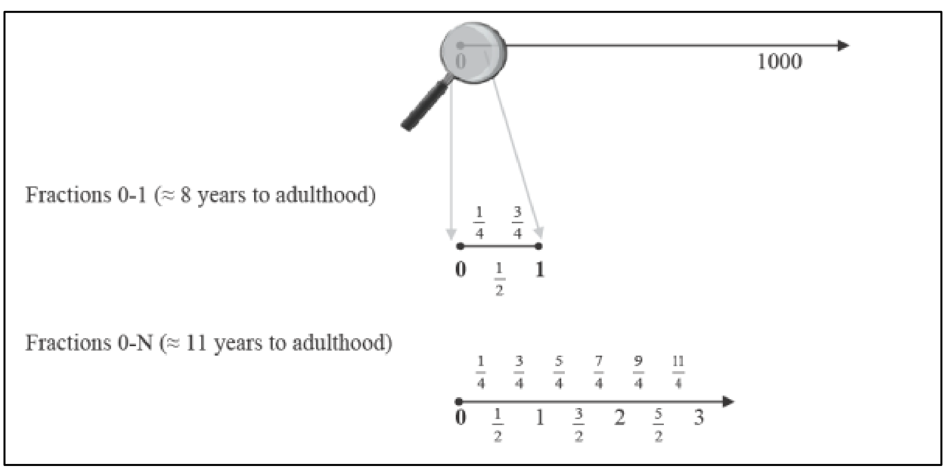


Note. From 'An integrative theory of numerical development,' by Siegler, R. S., & Lortie-Forgues, H. (2014), *Child Development Perspectives 8*(3), 144-150. Copyright Robert Siegler and Hugues Lortie-Forgues. Reprinted with permission.

The development of fraction magnitude is suggested to develop from approximately 8 years of age, where children start to represent and connect an understanding of the symbolic notation of proper fractions (those between 0 and 1), extending to fractions 0 to *n* at approximately 11 years and beyond, as represented in Figure 2.8.

Figure 2.8

Proposed Development of Fraction Magnitude Understanding



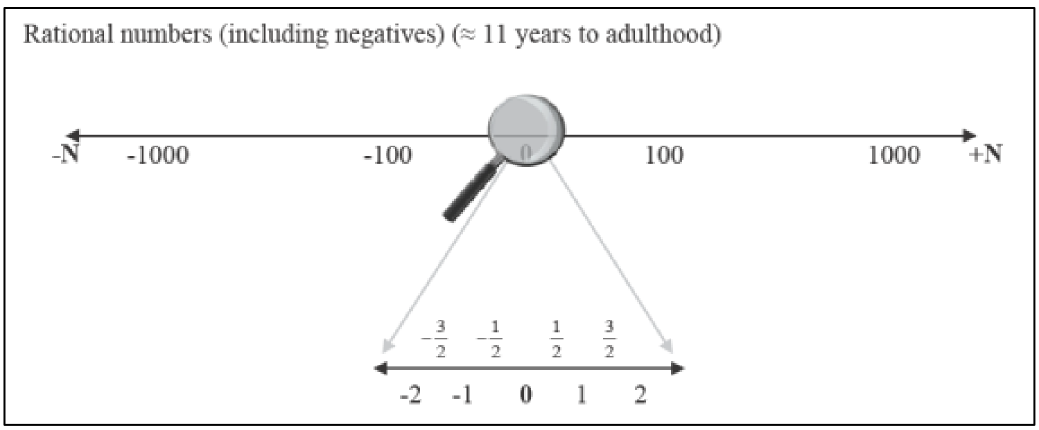
Note. From 'An integrative theory of numerical development,' by Siegler, R. S., & Lortie-Forgues, H. (2014), *Child Development Perspectives 8*(3), 144-150. Copyright Robert Siegler and Hugues Lortie-Forgues. Reprinted with permission.

2.3.1.4 Representing Accurately the Magnitudes of Rational Numbers

Finally, representing accurately the magnitudes of rational numbers, including fractions, decimals, percentages, and negative numbers (Siegler & Lortie-Forgues, 2014, p. 13) is said to develop from approximately 11 years of age though to adulthood. This development is represented in Figure 2.9.

Figure 2.9

Children's Proposed Conceptions of Rational Numbers to Adulthood



Note. From 'An integrative theory of numerical development,' by Siegler, R. S., & Lortie-Forgues, H. (2014), *Child Development Perspectives 8*(3), 144-150. Copyright Robert Siegler and Hugues Lortie-Forgues. Reprinted with permission.

The ITND suggests that while very young children can reliably discriminate between non-symbolic ratios form infancy, it is whole number knowledge that lays the foundation for fraction ideas to be conceptualised, from approximately 8 years of age. Research examining how this theory informs practice will now be discussed.

2.3.1.5 Implications for Teaching and Learning

The ITND recognises fractions as an inherently important part of numerical development and excluding or delaying them until middle and upper primary schooling is described as inadequate and unnecessary (Siegler et al., 2013). According to several studies (see Jordan et al., 2017; Siegler, 2016; Wang & Siegler, 2023), there is growing evidence that understanding magnitude is key to mathematics learning in all areas of the discipline. Siegler et al. (2011) justify their position further,

> If magnitudes are central to understanding fractions as well as whole numbers, then instruction that emphasises magnitude understanding is more likely to succeed than instruction that does not emphasise magnitude understanding (p. 293).

Further, the representation of magnitudes for whole numbers (e.g., Booth & Siegler, 2006, 2008; Halberda et al., 2008; Holloway & Ansari, 2008; Jordan et al., 2013; Sasanguie et al., 2013) and fractions (Bailey et al., 2012; Siegler & Pyke, 2013; Siegler et al., 2011, 2012) both predict overall mathematics achievement, providing a significant reason for instruction to be influenced by this theory of learning, specifically in the early years of education.

An essential part of this theory is that children learn and reason with many whole number properties common to rational numbers, such as they are countable, can be represented as a symbol, and possess commutativity for addition and multiplication. However, other whole number properties are not generalisable for all rational numbers. For example, fractions do not possess a unique successor like whole numbers do, because fractions can be represented by an infinite number of equivalent fractions. Moreover, multiplication does not always make larger, and division does not always make smaller when operating with fractions—unlike operating with whole numbers. Thus, learning when and how these properties apply is foundational to this theory (Keijzer & Terwel, 2002; Moss & Case, 1999).

In terms of applying this theory to a pedagogical approach, it appears this theory is largely dependent on the measurement meaning of fractions, because of the reference to representing and ordering whole number and fraction magnitudes using number lines. There is also the consideration of the spatial reasoning influence in this theoretical perspective, as it assumes that numerical magnitudes—both whole number and fractions—develop along a mental number line. This is typically described as a horizontally orientated line that one imagines smaller numbers to the left of and larger numbers to the right (Siegler & Braithwaite, 2017; Siegler & Opfer, 2003; Siegler et al., 2009). This suggest that there is an implicit link between spatial reasoning and early number development in terms of how numbers are proportionally considered in relation to one another.

2.3.2 The Ratio Processing System

The ratio processing system (RPS) suggests our cognitive architectures allow us to perceive quantity in non-symbolic ratios (Lewis et al., 2016; Matthews & Ziols, 2019). In other words, this theory suggests that from infancy, we are intuitively drawn to ratios that are presented in pictorial or concrete form. This theory relates to the proto-quantitative schemas explored by Singer and Resnick (1992) in the fraction as ratio meaning discussed above. Matthews and Ziols (2019) explain this perspective:

> We take the position that leveraging these proto-numerical intuitions to formalize a ‘sense’ or ‘feel’ for proportion may provide an alternate route to building rational number concepts (see also Abrahamson, 2012; Matthews & Ellis, 2018). This account is quite different from approaches positing that rational number concepts most naturally emerge from processes such as equipartitioning or learning to coordinate units (e.g., Hackenberg, 2007; Olive & Lobato, 2008; Pothier & Sawada, 1983; Steffe, 2001), (p. 215).

RPS theorists argue that children have cognitive architectures that support spatial ratio concepts from an early age and leveraging this perceptual sensitivity to non-symbolic ratio magnitudes can enable more flexible understandings of the different meanings of fractions (Lewis et al., 2016). For example, researchers have found children as young as 6-months of age are able to process the difference between non-symbolic ratios, which was also acknowledged in the theory previously discussed (se Cordes & Brannon, 2009; McCrink & Wynn, 2007; Siegler & Braithwaite, 2017). Other perspectives from the neuroscience field have supported the notion that magnitude coding—mapping number quantity to space (such as the mental number line)—is accessible by infants as young as 5-months old. For example, De Hevia and Spelke (2010) found children noticed discrepancies in ascending and descending line lengths when a predictable pattern of halving or doubling was not followed. Other studies have also yielded evidence for cross-dimensional transfer (e.g., from 2D map representation to a 3D physical context, which requires and appreciation of spatial proportion and ratio), suggesting that magnitude information regarding various dimensions is coded in one schema (De Hevia & Spelke, 2010; Möhring et al., 2014; Lourenco & Longo, 2009). This theory was evident in the way young children can engage with the various meanings of fractions (i.e., often through perceptual, spatial, non-symbolic contexts) described in the previous section, and the implications for teaching and learning will now be discussed.

2.3.2.1 Implications for Teaching and Learning

Foregrounding the RPS as an approach, focuses on perceptual abilities that enhance and allow for the exploration of fraction magnitudes (e.g., estimating and ordering fractions). This was evidenced by Spinillo and Bryant’s (1991) study examining children’s understanding of colour ratios (described in Section 2.2.2.5.1). Therefore, unlike the ITND, where the assumption is magnitude understanding of fractions develops *from* their whole number knowledge (despite early ratio discrimination), the RPS approach suggests engagement with, and accessibility to, whole numbers and fractions are on an equal footing (Matthews & Chesney, 2015). Although not evidenced in contemporary school curricula, the literature suggests young children's cognitive architecture is compatible with the development of non-symbolic, fraction as ratio meanings, as described by Singer-Freeman and Goswami's (2001) study of equivalent fraction analogies (half a pizza is proportionally equivalent to half a box of chocolates; see also Boyer & Levine, 2012; Duffy et al., 2005; Huttenlocher et al., 1999).

Based on the body of work underpinning the RPS, Matthews and Ziols (2019) suggest that this perceptually based intuition, for non-symbolic ratio and proportion magnitudes, should inform existing theory and help provide a basis for the design of more effective instruction for the development of rational number concepts. However, what is evident from the literature regarding the RPS is that studies seldom investigate this theoretical perspective from a classroom, intervention perspective in the early years of schooling. That is, the integration of the perceptual and spatial aspects of the non-symbolic fractional contexts are tested in assessment and experiment-based contexts and linked to participants' understanding of symbolic fractions and ratios, rather than developed in a real-world classroom environment during the teaching of fractions.

2.3.3 The Reorganisation Hypothesis

The early development of fraction understanding has been theorised by Olive (1999), Steffe (2001) and Steffe and Olive (2010) and labelled the *reorganisation hypothesis*. This theory has many similarities to the ITND, however it suggests that children's whole number knowledge serves as the foundation of and springboard for fraction knowledge (Biddlecomb, 2002; Norton & Hackenberg, 2010; Olive, 1999; Steffe, 2001; Steffe & Olive, 2010). Specifically, they suggest that children's integer counting schemes are reorganised to accommodate their fraction schemes, which are based on the unit fraction (Steffe, 2001). Steffe and Olive (2010) explain that 'a new scheme is constructed by using another scheme in a novel way, the new scheme can be regarded as a reorganisation of the prior scheme' (p. 1). Thus, a new scheme is developed and implemented when a child can use their existing knowledge and understanding to solve problems more efficiently and effectively.

This theory for fraction development states that children's initial partitioning, measurement and part-whole fraction schemes are based largely on children's knowledge of existing whole number concepts and operations. This includes, the cardinal and ordinal principles, and whole number operations such as addition (Norton & Hackenberg, 2010). As they accommodate the unit fraction, they iterate this unit to recreate the whole (e.g., $\frac{3}{7}$ is 3 iterations of $\frac{1}{7}$).

More recently, Tzur (2019) moved this reorganisation theory forward by describing the reorganisation of fractions as multiplicative relations. This perspective is grounded in the development of hypothetical learning trajectories from multiple studies (Simon & Dougherty, 2014; Norton & Boyce, 2013; Saenz-Ludlow, 1994) describing a progression of reorganisation of the iteration, measurement-based fraction schemes (Tzur, 2019).

2.3.3.1 Implications for Teaching and Learning

The reorganisation hypothesis suggests children's part-whole and measurement fraction schemas are developed from a reorganisation of their whole number development. The basis of this approach is an interpretation of partitioning, where the foundation is 0. That is, the unit fraction is determined as a single part of the whole and iterated to re-assemble the whole. There is agreement that the importance of establishing unit fraction understanding is critical for understanding fraction magnitude more broadly, however, the idea of the unit fraction as the unifying component between fractions and whole numbers is problematic. That is, while this hypothesis intends to describe multiplicative relationships, the process of iteration lends itself to repeated addition within measurement meaning, rather than exploring multiplicative structures (e.g., the generation of composite units) (Corley, 2013). Further, several researchers are sceptical about the problems that arise from such an emphasis on additive approaches (Lamon, 2001; Saenz-Ludlow, 1994; Streefland, 1991). Moreover, given the different interpretations and meanings of fractions, reliance on any single meaning is suggested as ineffective (Bruce et al., 2013; Lamon, 2012). Instead, other researchers (see Bruce et al, 2013; Confrey et al., 2014; Empson, 1999; Siemon, 2013) argue that a more appropriate emphasis should be the exploration of children's multiplicative partitioning schemas within multiple meanings of fractions, including ratio and operation, as this allows a greater development of partitive and iterative fraction ideas to develop concurrently.

2.3.4 The Splitting Conjecture

The splitting conjecture as a theoretical perspective to the development of fractions based on partitioning has been defined and explored by many researchers (see Confrey, 1994; Empson et al., 2006; Norton & Wilkins, 2012; Steffe, 2004). An examination of these perspective reveals that researchers have theorised this conjecture from one of two sperate, but related definitions, one driven by Confrey (1994) and one driven by Steffe (2004). Steffe's definition is based on an iterating and measurement approach which is derived from children's counting schemes, discussed above as the basis of the reorganisation hypothesis and, to a degree, the ITND. This discussion on the splitting conjecture will examine literature associated with Confrey's (1994) perspective, which is based on multiplicative partitioning perspective as summarised by Norton and Wilkins (2013):

> There are at least two independent but connected primitive constructs that lead to a robust understanding of numeration: one is counting, and the other is splitting. Splitting has its roots in activities like sharing, magnifying, shrinking, copying, and reproducing and is the primitive that leads to the development of multiplication, division, and ratio. There are fundamental, early, and essential ties between ratio and two-dimensional space that make a set-based approach to splitting inadequate, and necessitate careful ties to area, slope, rate, and similarity. (pp. 255–256)

Unlike Steffe's perspective of splitting (partitioning), Confrey's perspective considers splitting as multiplicative because the origin of the splitting context is '1'. That is, to determine a unit fraction, the whole is considered as the starting point for *n*-splits to be applied as repeated multiplication, as all parts created simultaneously (Confrey & Harel, 1994). As described above, in contrast, Steffe's (2010) perspective of splitting has an origin of '0', whereby a unit fraction is created from the whole and iterated to recreate the whole, emphasising repeated addition.

Critics of Confrey's (1994) interpretation of splitting state that this approach provides a didactical obstacle to learning fractions, which should be avoided (see Cortina et al., 2014; Tzur, 2007). That is, unlike epistemological obstacles (also described as cognitive conflict; Tall & Vinner, 1981) which unavoidably arise when the development of one mathematical idea interferes with the development of another (e.g., whole numbers and fractions); didactical obstacles are related to the materials, representations and procedures children are exposed to that enforce limited understandings or misconceptions of the new idea and should be avoided. Cortina et al. (2014) suggests that the splitting conjecture promotes such didactical obstacles because they believe it does not enable the reassembly of parts; it emphasises the part-whole meaning of fractions by considering the splits as separate parts to the whole and does not support fraction as ratio understanding. With regard to the last point, it appears they misinterpret the *many-to-one* idea as a foundation to the fraction as ratio meaning, by confusing this with the part-whole meaning. Their response to a splitting conjecture appears to be based on the ‘reinvention’ (Gravemeijer, 1994) of measurement, via iteration of fractions, rather than an appreciation of the unit fraction as a result of splitting.

In contrast to the view that rational number reasoning is developed upon a reorganisation of children’s whole number counting schemas (e.g., Cortina et al., 2014; Steffe, 2004; Tzur, 2019) or as the basis of whole number magnitude (e.g., Siegler et al., 2011), the rationale for the splitting conjecture is based on the foundation of children dividing and reassembling quantities, which is the foundation to multiplicative thinking. The greatest disparity to this theory and the previous is that it assumes that ‘splitting contrasts strongly with counting, the action on which a number system is most often built’ (Confrey & Smith, 1995, p. 70).

2.3.4.1 Implications for Teaching and Learning

To explore the splitting conjecture in more detail, Confrey’s work is examined more widely. Confrey et al. (2014b) developed a theoretical framework for rational number reasoning based on an extensive synthesis of over 500 studies. Seven conceptual sub-areas were identified for all rational number knowledge: (i) equipartitioning/splitting; (ii) multiplication and division; (iii) fractions; (iv) ratio and proportion; (v) length, area, and volume; (vi) similarity and scaling; and (vii) decimals and percentages (Confrey, 2008; Confrey et al., 2014a). After identifying the conceptual sub-areas from the corpus of research examined, a series of learning trajectories for each subgroup were proposed. The equipartitioning learning trajectory was empirically examined from a three-year longitudinal study for children Years 3-5 (see Confrey 2012; Confrey & Maloney 2010), however much of this framework is a conjectured map of the connections and sequences of ideas children may develop throughout primary school. These learning trajectories are presented in Figure 2.10.

Figure 2.10

Rational Number Reasoning Learning Trajectories (Confrey et al., 2014b).

[Image removed due to copyright restrictions].

Note. From ‘Equipartitioning, a foundation for Rational Number Reasoning’ (p. 69), in *Learning over time: Learning trajectories in mathematics education,* by A.P. Maloney, J. Confrey and K. H. Nguyen (Eds.), 2014, Information Age Publishing. Copyright 2014, by Information Age Publishing.

The learning trajectories shown in Figure 2.10 highlight the connections and possible pathways that children are conjectured to move through in the development of rational number ideas and concepts. However, it is worth noting that Confrey (2012) states, that this framework would be better represented as a cylinder, emphasising the connectedness between the ideas within the various rational number domains.

Confrey and colleagues (2014b) developed the learning trajectories for rational number reasoning, which are described as a *framework for understanding* rational number (Confrey et al., 2010; Sztajn et al., 2012). The foundation of this approach is that equipartitioning as their definition of splitting (also considered multiplicative partitioning; referred to as partitioning in this thesis) is viewed as the gateway to developing multiplication, division, ratio, and rate ideas in the domain of rational number reasoning (and fractions specifically). This view suggests that these domains cannot be separated into compartments that are both studied and taught in isolation (i.e., fractions versus whole numbers or fraction meanings viewed and taught as independent domains; Confrey, 1994). The basis of the rational number framework is that all of the learning trajectories are derived from the central, Equi-Partitioning Learning Trajectory (EPLT; Confrey et al., 2014b; Confrey & Maloney, 2010) that is developed on the basis of the fair share idea.

What is unique to this theory for rational numbers is the interconnectedness represented within and between the individual learning trajectories. As highlighted in Figure 2.10, this series of rational number ideas within the learning trajectories depict the typical ordered pathways of understanding children develop, rather than just accumulating knowledge as discrete topics. The framework represents a progression of learning that develops in the complexity of ideas over the primary years of schooling.

Another important element from this research is how the five typically meanings of fractions (fraction as part-whole, measure, operator, quotient, and ratio), first identified by Kieren (1976) and elaborated on by Behr et al. (1983), is simplified into three meanings. The three meanings are *fractions as a relation* (ratio and rate), *fraction as an operator* and *fraction as a measure* and are intended to be taught in relation to one another. Figure 2.11 has been adapted to show the how the three meanings of fractions are represented in this framework.

Figure 2.11

Map of Rational Number Concepts Grouped in the Three Meanings of Fractions (Confrey et al., 2014b)

[Image removed due to copyright restrictions].

Note. From ‘Equipartitioning, a foundation for Rational Number Reasoning’ (p. 69), in *Learning over time: Learning trajectories in mathematics education*, by A.P. Maloney, J. Confrey and K. H. Nguyen (Eds.), 2014, Information Age Publishing. Copyright 2014, by Information Age Publishing.

Confrey and Maloney (2010) simplify the definitions of each of the three meanings of fractions, with reference to the foundational ideas presented in Figure 2.11 above:

- $\boldsymbol{\frac{a}{b}}$ **as a Relation:** Ratio, through two-dimensional ‘many-to-one’ numerical relationships, ratios unit and unit ratios. For example, many-to-one is an indication of ratio units, whereby children may identify there needs to be the same number of flowers (*many*) for each vase (*one*). A unit ratio is the understanding that 12 flowers and three vases versus 15 flowers and five vases still results in the unit ratio of three flowers for each vase (*proto-ratio*).
- $\boldsymbol{\frac{a}{b}}$ **as an Operator:** Through the act of *fair sharing,* and naming ‘1-nth-of...’ and ‘n (times) as many’ the referent unit resulting from equipartitioning (p. 973). Fraction as an operator from this perspective also includes the fraction as quotient meaning identified above, because of the focus on partitive division and recursive multiplication in sharing *n* objects between *m* people.
- $\boldsymbol{\frac{a}{b}}$ **as a Measure:** This fraction meaning starts with the ideas that many (objects) can be named as one meaning that any object (number) can be measured by an infinite

number of parts (fractions), derived from fair sharing. That is, if I were sharing 12 flowers between three vases, four flowers are the size of each fair share created from the collection of flowers. If I were to share a cake, I could cut it into any number of equal parts that represent a measure of that cake, whereby the relative size of the cake remains unchanged.

In their research on rational number reasoning, Confrey et al. (2014b) identify a range of cognitive behaviours that are essential for interpreting the children's interaction between the various ideas within each of the fraction meanings (see Table 2.2).

Table 2.2

Summary of Cognitive Behaviours (Paraphrased from Confrey, 2012, p. 159)

Strategies are selected and employed by the child to solve problems and are typically seen at the lower levels. The strategy is merely employed as a means to try to solve the task (e.g., dealing a set of objects one by one to share the collection; staking/ matching groups of objects initially to check for equality).
Mathematical reasoning practices are used to explain strategies and solutions. These include *naming* and justifying as well as providing 'proof' such as physically stacking shares of coins to establish equality of each share.
Emergent properties and relations of the mathematical ideas of focus act as 'localised' generalisations and guide future strategy choice as the student coordinates strategies and reasoning practices. The emergent properties and relations cognitive behaviour suggest the child is anticipating solutions and proposing regularities in approaches (i.e., *if more people share, then each share is smaller*).
Systematic tendencies towards certain errors or alternative conceptions, such as cutting a circle horizontally rather than with radial cuts to create fair shares, and they need to be addressed with the child.

Generalisations of increasing power slowly emerge as the understanding of the three cases for partitioning merge:

- Case A: sharing a collection of *m/n* objects fairly where m and n are natural numbers
- Case B: sharing a single whole among n children
- Case C: Sharing multiple wholes *m* among *n* children where *n* is not a factor of *m* (includes proper and improper fractions).

With reference to the ideas explored in the early years of this framework, Confrey (2012) states that strategies are an important element of understanding how young children come to understand how fractions are created and operated on. For example,

> …folding a piece of paper is different from cutting or marking it, and the different types of activities produce different insights …with cutting it is not uncommon to witness children using the congruence of the parts by stacking one piece on top of the other. When folding, congruence is built directly into the activity through symmetries, but the result of the action is hidden until the paper is unfolded, providing opportunities to examine one's predictions. (Confrey, 2012, p. 167)

This description of the strategies such as folding, visualising, predicting, and examining the geometric and measurement properties of a simple piece of paper are clearly connected to spatial rather than symbolic or numerical understandings in the early years, which is not acknowledged explicitly in the present or previous theoretical perspectives.

Also noteworthy is that the description of the ideas in Figure 2.11 related to approximately the first three years of schooling suggests, if only implicitly, the spatialised contexts in which children might engage with during the exploration of these ideas, reflected in such ideas as *scaling, geometric symmetries, length,* and *area* as examples. For example, in the region where Year 1 and 2 children would be exploring the idea of *partitioning a single whole*, it is also connected to spatial-based ideas such as *geometric symmetries*, *conservation of the whole*, *conservation of length*, *area* and *scaling*, to name a few. This suggests that children need to have experiences that promote their spatial reasoning capabilities to effectively develop their early understanding of an extended range of fraction meanings.

Bruce et al.'s (2015a) research, resulting in the *Fractions Learning Pathways* curriculum tool for Ontario is an example of how a splitting approach can be effectively used as a basis to explore and develop fraction understanding, This research-informed work was inspired by Confrey's splitting perspective and provides a suite of field-tested tasks that guide teachers in the teaching and learning of fractions from approximately Year 1 - 10 and includes some of these more 'spatial' ideas within the suggested fraction activities. Significant gains in children's learning of fractions have been reported in the middle and upper primary years from this research. However, the limited literature relating to the early ideas and fraction meanings articulated in Confrey et al.'s (2014b) framework above, has not explicitly addressed the connection to spatial reasoning, meaning there is great potential to explore this avenue in the early years.

2.3.5 Discussion of Theoretical Perspectives

Two key issues have emerged from the analysis of the theoretical perspectives discussed on the development of rational number knowledge. The first is that there are multiple meanings of fractions that children need to develop and understanding of in relation to each other to mitigate the misconceptions and difficulties they present in later schooling in this area of mathematics. The second is that the way in which young children engage with a range of early fraction ideas suggests that spatial reasoning is an implicit yet seemingly underutilised cognitive ability in the development of this area of mathematics.

With regard to the first issue, Thompson and Saldanha (2003) take a strong position in describing how Kieren and the Rational Number Project's work has been largely interpreted and implemented:

> Our feeling is that their attempt to map systems of complementary meanings into the formal mathematical system of rational numbers will necessarily be unsatisfactory in regard to designing instruction for an integrative understanding of fractions…A variety of sources suggest it is through the development of a web of meanings that entails conceptualisations of measurement, multiplication, division, and fractions. We emphasize conceptualisations of measurement, multiplication, division, and fractions. This is not the same as measuring, multiplying, and dividing. The latter are activities. The former are images of what one makes through doing them. (pp. 14–15)

This point speaks to the present research, in that while Kieren and Behr et al.'s extensive body of work is being not criticised, it does not currently serve the purpose of integrating such meanings within the broader theoretical domain of rational number in the early years. That is, identifying and working with the various meanings of fractions is not sufficient based on connecting the three concepts of partitioning, unitising, and quantitative equivalence, as there needs to be a connection between the bigger mathematical ideas of measurement, multiplication, and division in addition to fractions. It seems that Confrey et al.'s (2014b) framework for rational number reasoning appreciates this complexity of rational number theory that has both theoretical and pedagogical rigour, because it is founded on the premise that division and multiplication are developed from multiplicative partitioning contexts and coordinated with, not derived from, counting, addition and subtraction (Confrey, 2008). This premise is supported by the literature that early proportional awareness of non-symbolic ratios is inherently intertwined within children's mathematical development from infancy—which is acknowledge in the ITND, even though counting and whole number knowledge was deemed the basis of fraction understanding. Although the literature describing young children's proportional and non-symbolic capabilities with fraction context largely comprises experimental or assessment-based studies, it does suggest there is a gap in how these early perceptual abilities can be harnessed to promote young children's understanding of the fraction meanings.

This leads to the second issue identified from this theoretical review, in that the role of spatial reasoning explicitly is missing from the discussion about young children's engagement with various fraction meanings and their underpinning ideas—regardless of the foundation in which the theory resides. For example, the prevalence of perceptual and spatial skills to perceive spatial ratios and magnitudes is evident across the differing theories as identified above. The early sensitivity to non-symbolic ratios and Confrey's description of folding and cutting when children are exploring partitioning also suggest this engagement relates to the development of mental images of quantity and magnitude, created through engaging in different spatial reasoning constructs. For example, young children's ability to distinguish between proportional and a distribution situation, such as in Spinillo and Bryant's (1991) study, draws on spatial aspects of geometry, measurement (length, area, etc.) as well as proportion (half). These ideas are also represented in Confrey et al.'s (2014b) learning trajectories—yet the spatial reasoning foundation is not explicit in the theoretical discussion of children's fraction capabilities.

Given this context, it seems critical to explore these early understandings and how spatial reasoning may affect children's development of fraction meanings and the associated concepts. Based on this rationale, spatial reasoning and its relationship to fraction development will now be discussed.

2.4 Spatial Reasoning and its Influence: Rational Number Reasoning

Before exploring the role spatial reasoning may play in developing early fraction concepts and ideas, it is helpful to define what is meant by the term. There are many interpretations of what spatial reasoning is, depending on the theoretical and disciplinary orientation of the definition (Bruce et al., 2015b).

2.4.1 Defining Spatial Reasoning

Battista (2007) explains that spatial reasoning 'is the ability to "see" [in one's mind], inspect, and reflect on spatial objects, images, relationships, and transformations' (p. 843). The Ontario Ministry of Education (2014) elaborates on this definition to provide contexts and situations relevant to children that help define this ability:

> [Spatial reasoning] involves understanding relationships within and between spatial structures and, through a wide variety of possible representations…When a child rotates a rectangular prism to fit into the castle she is building she is employing spatial reasoning, as is the student who uses a diagram of a rectangle to prove that the formula for finding the area of a triangle is ½b*h. Spatial reasoning vitally informs our ability to investigate and solve problems, especially non-routine or novel problems, in mathematics'. (p. 3)

Spatial reasoning is, therefore, an overarching term for spatial concepts, processes, and tools that learners engage with when processing a range of information, problem contexts and data sources and includes a range of spatial skills (Lowrie et al., 2021; National Research Council [NRC], 2006). Whiteley et al., (2015) describe spatial reasoning in the form of Tahta's (1989) 'powers' involved with working with space. These powers are:

- *Imagining*—also described as the process of 'seeing' what is said or presented.

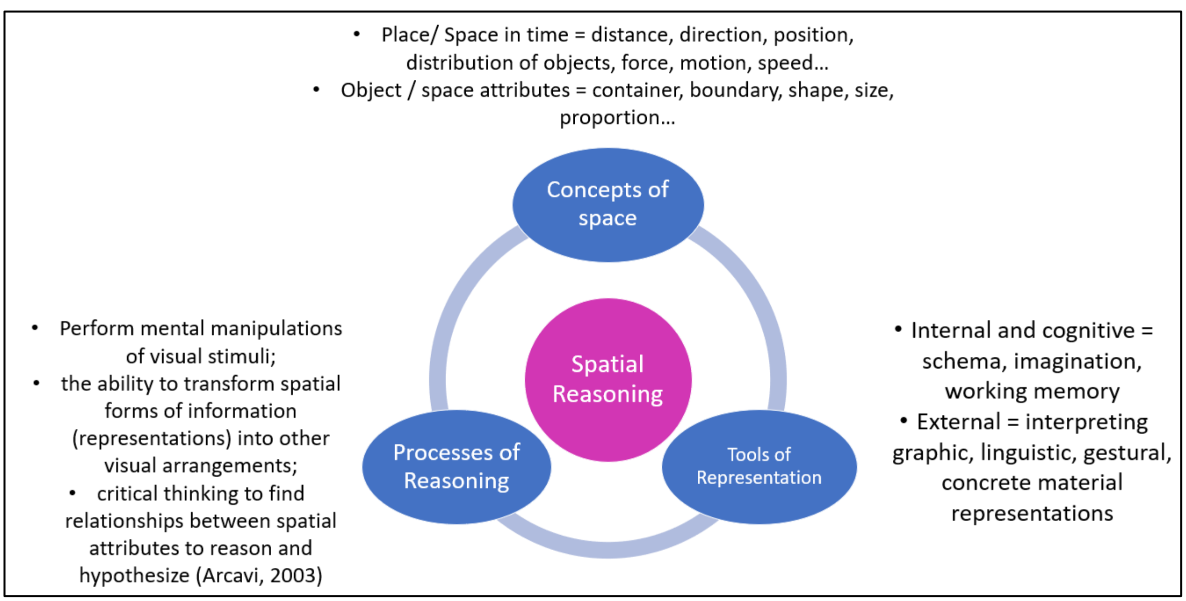
- *Constructing*—the process of conceptualising and seeing what is drawn and saying or communicating what is seen.
- *Figuring*—which involves drawing what is seen.

Whiteley et al., (2015) offers an extension to Tahta's framework to further refine our understanding of spatial reasoning and the processes and skills that constitute such learning. Although described as preliminary, the following list of verbs offers a characterisation of what spatial reasoning may entail in mathematics education: locating, orienting, decomposing, recomposing, shifting dimensions, balancing, diagramming, symmetrising, navigating, transforming, comparing, scaling, sensing, and visualising.

In a similar vein to Tahta's (1989) initial three 'powers', the NRC (2006) synthesised three core elements that describe spatial reasoning. This framework was developed from a synthesis of research from a wide range of disciplines, such as astronomy, education, geography, the geosciences, and psychology (NRC, 2006) and is recreated in Figure 2.12.

Figure 2.12

An Interpretation of the NRC's Spatial Reasoning Framework



The three elements within the framework are concepts of space, tools of representation and processes of reasoning. Concepts of space provide the conceptual and analytical framework within which data can be integrated, related, and structured into a whole (NRC, 2006). This element relates to space in time, object/field, and orientation or place ideas. It involves identifying the space relative to an object, its container, boundary, shape, and texture. This element includes an awareness of the spatial relations, including static spatial relations such as location (distance, direction, distribution) and connection, and dynamic spatial relations like motion, flow, force, intersection, and collision. Tools of representations are considered internal and cognitive, or external and graphic, linguistic gestural and provide the forms within which structured information can be stored, analysed, comprehended, and communicated to others (NRC, 2006). Representations are important to all areas of mathematics, so to have a framework that helps to articulate the internal and external components in relation to fractions from a spatial perspective is most useful. Representations will be expanded on later in this chapter due to their importance to the teaching and learning of mathematics. Processes of reasoning is described as the capacity to recognise and perform mental manipulations of visual stimuli; the ability to transform spatial forms of information (representations) into other visual arrangements; an awareness of the structural features of spatial information (e.g., identifying an ABC unit structure of a pattern) or objects (such as scale, orientation, perspective and proportion); and critical thinking to find relationships, reason and hypothesise to solve problems (Arcavi, 2003; Mulligan et al., 2018; NRC, 2006).

It is clear from these descriptions that there is no standard definition, and that spatial reasoning is often used interchangeably with spatial thinking, spatial abilities, and visual-spatial reasoning. However, Lowrie et al., (2018) suggest that in an instructional context, all three elements of the NRC's (2006) framework are considered and promoted in the effective teaching of spatial thinking. Therefore, this study will consider spatial reasoning as the relationship between an appreciation of space, the internal and external representations of this space, and the cognitive processes that enable children to reason and justify their thinking.

2.4.2 Spatial Reasoning and Fractions

With regard to the aforementioned discussion on spatial reasoning and the insights into children's non-symbolic, perceptual awareness discussed in the theoretical discussion above, one key construct emerged as dominant: *spatial proportional reasoning*. This was identified throughout several theoretical perspectives as the way in which young children demonstrate a sensitivity to early ratio and non-symbolic equivalent quantities.

As I examined the literature further in relation to spatial reasoning more broadly, the process of mental manipulations in children's perceptual awareness of partitioning, fair sharing and estimating changes in magnitude suggested that spatial visualisation was a relevant construct in the development of young children's rational number reasoning. These spatial constructs and their relationship to young children's development of fraction understanding will now be explored.

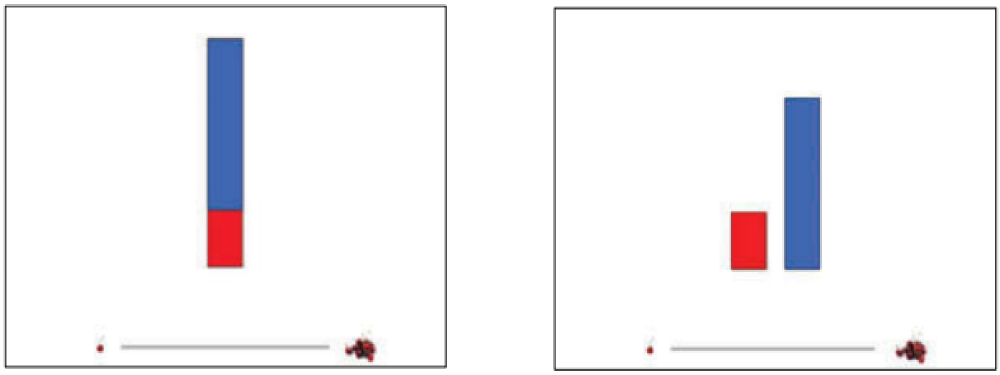
2.4.2.1 Spatial Proportional Reasoning and Fractions

Spatial proportional reasoning is described as the ability to reason about non-symbolic, relative quantities (Möhring et al., 2015) and, therefore, is associated with the concept of quantitative equivalence. Similar to this construct, *spatial scaling* is the process of transforming non-symbolic quantities while conserving relational properties, and it is therefore an important aspect of spatial proportional reasoning (Barth et al., 2009; Boyer & Levine, 2012; McCrink & Spelke, 2010). Several researchers have demonstrated that spatial proportional reasoning and spatial scaling are closely related cognitive processes (Begolli et al., 2020; Boyer & Levine, 2012; Möhring et al., 2015). The present study is not concerned with differentiating between spatial scaling and spatial proportional reasoning from a cognitive perspective; therefore, for the purposes of this thesis, spatial proportional reasoning will imply both proportional and scaling capabilities.

Spatial proportional reasoning has been examined in relation to young learners' fraction development (Boyer et al., 2008; Boyer & Levine, 2012; Möhring et al., 2015). For example, Möhring et al. (2016) found that 8 – 10-year-old children's formal fraction knowledge was influenced by their ability to use spatial scaling to reason in non-numerical, proportional reasoning situations. The task used in that study was a cherry juice mixture context, first employed by Boyer and Levine (2012). In this task, children were presented with either one or two vertical columns partitioned into two parts (see Figure 2.13). The red portion represented the cherry juice and the blue portion represented water. Underneath each column was a horizontal scale with one cherry to the absolute left, indicating a 'weak'-tasting mixture, and a bunch of cherries to the absolute right, representing a 'strong'-tasting mixture. No numerical information was presented on the horizontal scale or the vertical juice and water column. Children were asked to position a peg on the horizontal scale to correspond to how strong or weak they thought the represented cherry juice and water mixture would taste, presented in a 'stacked' and 'side-by-side' context as presented in Figure 2.13.

Figure 2.13

Cherry Juice Example (Möhring et al., 2016)



Note. Examples of a stacked (left) and a side-by-side (right) presentation of cherry juice (e.g., six units) and water (e.g., 24 units) in the proportional reasoning task. From 'Spatial proportional reasoning is associated with formal knowledge about fractions', *Journal of Cognition and Development*, *17*(1), 67–84, by W. Möhring, N. S. Newcombe, S. C. Levine and A. Frick, 2016. Copyright 2021 by Taylor & Francis. Reprinted with permission.

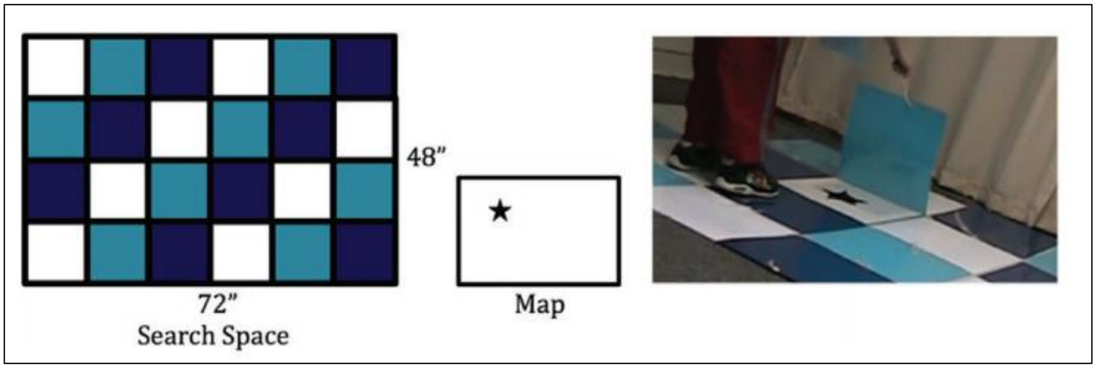
That study revealed that children's accuracy in judging the proportions were higher in the part-whole (stacked condition) as opposed to the part-part condition, hypothesised to be because in the stacked condition, it was easier for children to mentally align the proportional amounts to the rating scale (Möhring et al., 2016). However, they were not able to determine whether young children's previous fraction understanding impacted their ability to complete these spatial proportional reasoning tasks effectively, or whether their spatial proportional reasoning promoted their fraction knowledge. Despite these differences, the study suggests that young children are much more capable of engaging in these fraction and relation ideas than previously thought, which is consistent with Spinillo and Bryant's (1991) discussion on young children's early ratio understandings (see Section 2.2.5). These findings suggest that those children who demonstrate a better understanding of relative proportion may be able to better visualise fractions in terms of spatial analogies, which, in turn, may help them to understand numerical fractions, perhaps because they are more able to differentiate plausible and implausible answers (Möhring et al., 2016). These findings provide further motivation for the present study.

Spatial proportional reasoning relates closely to the fraction as an operator and fraction as a measure meanings because they support the ‘stretcher shrinker’ ideas associated with the operator meaning (such as double and half, times-as-many ideas) and the equivalent fraction idea underpinning the measurement meaning. However, these fraction ideas are considered to develop much later in children’s mathematical development (Huttenlocher et al., 1999; Piaget & Inhelder, 1967).

Evidence suggests that children from as young as 3 years of age can intuitively explore early fraction concepts such as partitioning, equivalence and unitising in proportional contexts at a perceptual level, utilising scaling capabilities (Frick & Newcombe, 2012; Huttenlocher, et al., 1999; Möhring et al., 2014; Vasilyeva & Huttenlocher, 2004). This context is exemplified by Jirout et al.’s (2018) search game (see Figure 2.14). In this study, 3 – 8-year-old children were required to locate an object hidden in a physical space (e.g., floor mat) using a simple map that indicated a nominated tile was in the top left-hand corner of the search space from the child’s perspective, as presented in Figure 2.14.

Figure 2.14

Spatial Scaling Search Game Materials



Note. From 'Scaling up spatial development: A closer look at children's scaling ability and its relation to number knowledge', *Mind, Brain, and Education, 12*(3), 110–119, by J. J. Jirout, C. A. Holmes, K. A. Ramsook and N. S. Newcombe, 2018. Copyright 2018 by John Wiley and Sons. Reprinted with permission.

The map was a much simpler, 2D version of the space the object was hidden within, and children had to use their partitioning and scaling skills to identify and locate where they thought the object was hidden. Although small numbers of children were involved, most children were successful, as they were able to understand how the proportions and locations of the 2D representation would translate (i.e., increase in scale) in the physical representation. This illustrates that through the identification and application of unit size and magnitude (mentally partitioning the map to determine where the star may be located in the 3D world), scaling is connected to the concepts partitioning and equivalence.

This skill requires an understanding of how distances in different-sized spaces are related, in addition to geometric correspondence (Downs, 1985; Newcombe & Huttenlocher, 2003; Möhring et al., 2014). Geometric correspondence is the ability to encode distances with some unit of measure, which is indicative of proportional understandings whereby the discrimination of space, time, number, and speed may be required (Brannon et al., 2006; Brannon et al., 2007; Möhring et al., 2012; Xu & Spelke, 2000). The learner mentally shrinks or expands spatial information (e.g., enlarging or shrinking an image), also described as the activity of 'zooming in' or 'zooming out', which internally transforms the magnitude information (Möhring et al., 2018).

Evidence provided by Frick and Newcombe (2012) also illustrates children as young as 5 years old can work with relative distances, using scales of 1:2 and 1:4, in solving location tasks using 2D maps and corresponding representations. Further evidence from Frick and Newcombe (2012) and Vasilyeva and Huttenlocher (2004) reveal that there is a great deal of development in the accuracy of children's spatial proportional reasoning capability between the ages of 3 – 6 years. Gilligan et al. (2018) found that children between the ages of 5 – 8 years achieve further gains in spatial scaling skills during this period, particularly in the flexibility and accuracy of their abilities.

The evidence from these studies suggests the early years provides an important opportunity to develop children's understanding of fraction ideas utilising this spatial construct. This is due to the connection between visualising how different spaces and objects can be partitioned without necessarily quantifying the measures; it also enables the child to visually compare quantities of the objects and shapes explored, to identify the different size of the parts created within the same objects. Further, it enables children to develop an understanding of proportional relationships between different objects and spaces, such as a map and its real-world space.

The literature presented in this discussion provides evidence for the links between the development of spatial proportional ideas and the proposition that this may positively influence children's development of formal fractional knowledge—specifically fraction as an operator and fraction as a measure understandings. It is imperative to note, however, that in each of these examples, while children (especially under the age of 7 years) are demonstrating the ability to reason in some proportional contexts, it is not claimed that children of these ages can perform proportional computations or have a sophisticated multiplicative understanding required for proportional reasoning (He et al., 2018; Frick & Newcombe, 2012). However, this literature indicates that during the period of cognitive development between 5 – 8 years of age, an alternative approach to teaching fractions that encompasses spatial proportional reasoning may positively affect young children's understanding of early fraction concepts.

2.4.2.2 Spatial Visualisation and Fractions

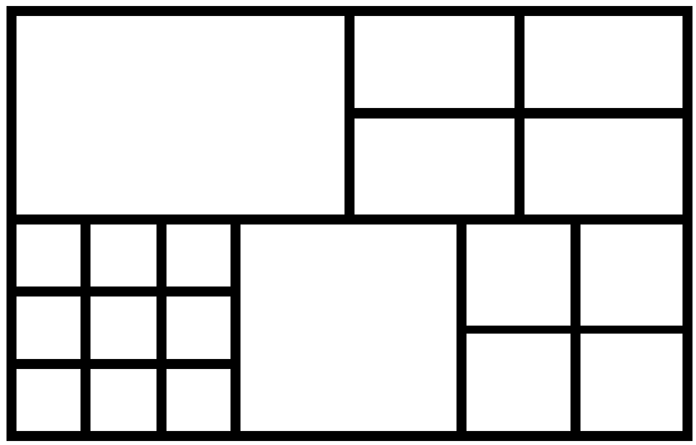
Spatial visualisation is considered one of the most complex categories to define under the umbrella of spatial reasoning. The difficulty lies in the fact that the term is defined in many ways. Linn and Peterson (1985) define spatial visualisation as a multi-step manipulation of objects. Many mental rotations and transformations may occur while the participant keeps a mental record of each application and its impact on the original image at hand. As a theoretically correlated construct of spatial visualisation (Linn & Petersen, 1985; Maeda & Yoon, 2013), *mental rotation* is the ability to imagine how an object would look if it were rotated; that is, to mentally turn a 2D or 3D object (Frick et al., 2013). However, some researchers (e.g., Hawes et al., 2019; Lowrie & Logan, 2023) acknowledge that spatial visualisation is a complex construct to define and separate from other constructs, such as mental rotation and mental transformation. As described above, the present study is not concerned with differentiating between what constitutes the nuances between closely related spatial constructs; therefore, for the purposes of this thesis, the term 'spatial visualisation' will imply mental rotation and mental transformation, unless otherwise noted.

A common example of spatial visualisation is imagining folding a piece of paper several times, whereby the process of each transformation needs to be remembered as well as visualising the outcome of each fold on the paper. This has important links with early years fraction instruction as paper folding is a common task that is introduced to help children develop ideas about equality and partitioning a whole. Other definitions of spatial visualisation include describing creation of a mental image and naming the visual and or spatial information of the object in one's mind, in addition to performing several 'imagistic transformations' or manipulating the object mentally in some way, without regard for the speed of the solution (Carroll, 1993; Höffler, 2010; Lohman, 1988; McGee, 1979; Sorby, 1999). Lowrie et al., (2016) provide a succinct definition which described spatial visualisation as the ability or skill drawn upon to mentally transform or manipulate spatial properties of an object/image.

Mix et al. (2016) found that spatial visualisation competency is a strong predictor of general mathematical performance across multiple grade levels. Lamon (2001) contends that spatial visualisation and the concepts of partitioning, unitising and quantitative equivalence form a symbiotic relationship. That is, there is a close relationship between the ability to visualise or predict the outcome of each concept, in that they each are considered to rely on the generation and manipulation of mental imagery in some way. For example, children need to visualise the decrease and increase in the size of parts when considering how many shares are created. They need a visual awareness of the how parts can be renamed to identify equivalent fractions. This relationship is exemplified in the following task cited by Lamon (2012; see Figure 2.15), which can be used to explore both fraction as a measure and fraction as an operator meanings in the early years of primary school.

Figure 2.15

The Stimulus for Problem: Where Can You See $\frac{1}{8}$*?*



Note. From *Teaching fractions and ratios for understanding*, by S. J. Lamon, 2012. Copyright 2012 by Routledge. Reprinted with permission.

In this task, a child is invited to use spatial visualisation to perform mental manipulations of the visual stimuli to imagine other arrangements of the parts to determine different fractions. Here, the child may mentally partition (by repeated acts of halving), to describe and reason where they can see different fraction quantities—such as double and half, times-as-many from the operator meaning, and explore different unit fractions and their equivalencies from the fraction as a measure meaning. Explicitly emphasising the use of spatial visualisation when partitioning a whole such as the one in Figure 2.14 can also help young children discover and conceptualise the idea that the magnitude of each identified part (unit) is reduced as the number of parts increases (Kieren, 1993; Siemon, 2003), because, much like the paper folding example described above, they are visualising the outcome of repeated partitioning and how this reduces the size of each part.

In terms of literature concerning a pedagogical emphasis on spatial visualisation, Bruce et al. (2015) developed and assessed the efficacy of a 'spatialised' curriculum in Years 1 to 3, focusing on static and dynamic symmetry, geometric congruence, and transformations, including mental rotation, which are all components of spatial visualisation. Although this program did not have an explicit number or fraction focus per se, the program did allow children to focus on the unit structures and part-part relationships through the various geometric contexts, evoking and developing children's mental manipulations and imagination skills (Bruce et al., 2015b). For example, children were scaffolded to imagine dynamic movement of shapes and object through visualising symmetrical parts—which could connect to children's understanding about part-whole and halving ideas. In addition, this program revealed significant growth in children's spatial language, geometric reasoning and mental rotation abilities and substantially affected children's numerical comparison skills (Bruce & Hawes, 2015). These findings suggest that the development of mental manipulation skills through visualising different geometric (and, therefore, spatial) contexts may also have substantial benefits for partitioning and unitising (such as disembedding, manipulating and transforming objects mentally). This suggests that spatial visualisation may provide a cognitive pathway for the development of early fraction ideas, which is worthy of further exploration as a pedagogical approach to this area of mathematics.

2.5 Representations, Spatial Reasoning and Fraction Development

Representations are fundamental to the teaching and learning of mathematics as they support problem-solving, conjecturing and the communication of ideas and concepts (Goldin & Janvier, 1998). Further, as described in Section 2.4, tools of representation (NRC, 2006) are a key element in the spatial reasoning framework used to define and describe how different spatial constructs are enacted and utilised in learning. Therefore, a review of the role of representations in relation to fractions and spatial reasoning is required to understand how these may influence understanding in the early years. Goldin and Shteingold's (2001) extensive work states that representations are typically categorised into two forms: internal and external. These forms of representations will now be explored.

2.5.1 Internal Representations

Internal representations are the creation and description of the psychological mathematical systems of individuals (Goldin & Shteingold, 2001); also considered as idiosyncratic, ideas, constructs and images that are created, held, and manipulated in one's mind (Lowrie, 2010; Kosslyn, 1988; Presmeg, 1986/2006). They are the essentially the networks or architectures of our thoughts and mental images that we create from our experiences, which we remember and use as a basis of knowledge.

Goldin (1998) describes these in the form of systems:

- *verbal/syntactical representations*, which described the way a learner processes language and understands this internally to communicate with externally, such as knowing 'half' as a quantity within different contexts
- *imagistic or mental images* that include visual and spatial cognitive configurations, such as imagining what a quantity may look like or imagining the process of partitioning and objects in one's mind
- *formal notational representations*, such as performing mental arithmetic operations
- *strategic and heuristic processes* that involve students mentally organising a problem and mapping the process for problem-solving.

Importantly, Goldin and Shteingold (2001) note that while internal representations can be theorised in such a way, we cannot, of course, view any person's internal representations directly.

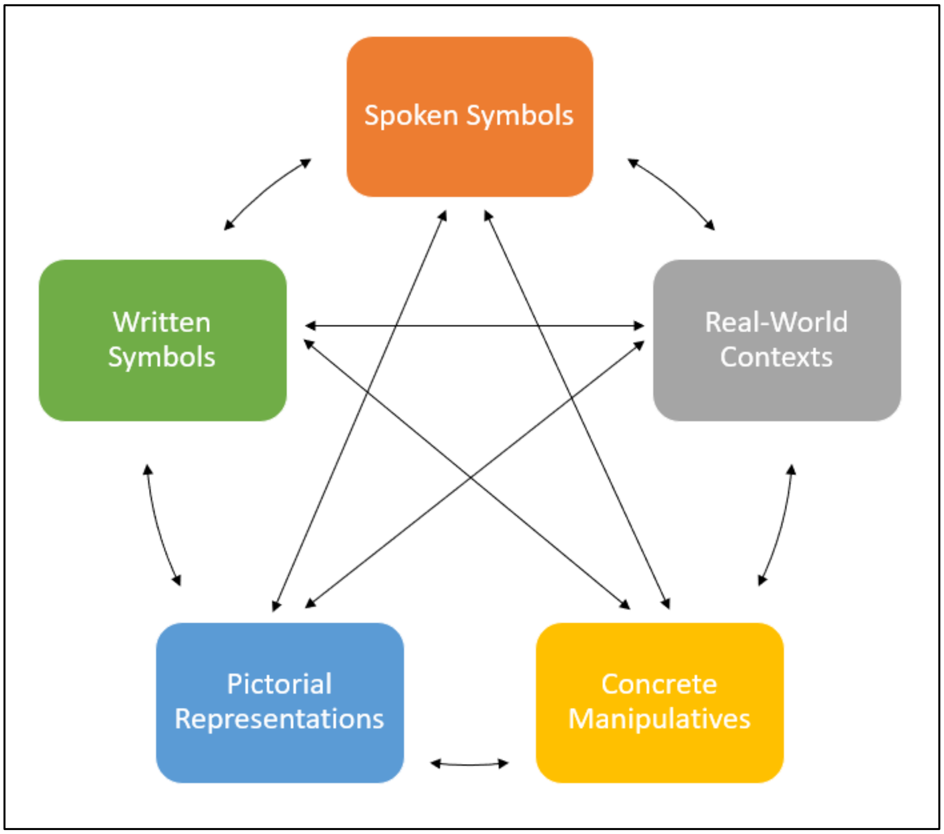
Instead, they act as ‘abstractions of mathematical ideas or cognitive schemata that are developed by a learner through experience’ (Pape & Tchoshanov, 2001, p. 119) that are not always intuitively evoked by children but are developed and built up over time through their mathematical experiences (Goldin, 1998). Thus, each child forms their internal representational system, which we can only make inferences about based on their interaction and discourse associated with their external representations. Regarding internal representations and fractions, the most relevant to the present study seems to be a connection between the spatial processes associated with spatial visualisation and spatial proportional reasoning, in the way children are imagining, manipulating, and organising objects in their mind, and how verbal/syntactical representations may contribute to revealing or indicating these transformations. The relationship between spatial reasoning and internal representations suggests that Goldin’s (1998) systems connect to the NRC’s (2006) element of concepts of space in regard to helping children visualise the size of an object, the size of its parts, how an object is partitioned or arranged (manipulated) and how the child names or ‘labels’ the parts.

2.5.2 External Representations

External representations help communicate and understand mathematical information, ideas, and concepts (Janvier et al., 1993). With regard to a focus on spatial reasoning, examining the way children use and engage with a range of external representations helps to interpret how they are building mental models of such ideas. Lesh et al. (1987) offer a model to explain the role of external representations in mathematical development and their connection to various concepts and problem-solving, called the ‘translation model’. This model is a reconceptualisation of Bruner’s (1966) Concrete–Representation–Abstract model and has been a focus in many of the large-scale research projects pertaining to children’s development of fraction concepts (e.g., the Rational Number Project: Behr et al., 1981; Cramer et al., 1997). This model has five modes of representations (in no particular order): 1) spoken symbols, 2) real-world contexts, 3) concrete materials (manipulatives), 4) pictorial representations and 5) written symbols. As shown in Figure 2.16, the intention of the model is that children develop their understanding within each of the five modes and the ability to move between the modes when representing a mathematical concept or idea (Lesh et al., 2003).

Figure 2.16

The Translation Model of External Representations



Note. Diagrammatic representation of the translation model. Adapted from 'Representations and translations among representation in mathematics learning and problem-solving', by R. Lesh, T. Post and M. Behr, in *Problems of representation in the teaching and learning of mathematics*, pp. 33–40, by C. Janvier (Ed.), 1987, Lawrence Erlbaum.

Given the widespread recognition of this model in mathematics education, it has been chosen to explore literature concerning fractions understanding. Each of these modes of external representations will be considered.

2.5.2.1 Spoken Symbols

Language use in mathematics as an external representation is not merely the process of matching a word to an indicated object (Kieren, 1999). Instead, the use of language, considered as spoken symbols, is an integration between the child's thoughts and actions as observed as constructive processes (such as partitioning/splitting, unitising, and re-unitising) in children's fraction understanding (Behr et al., 1992; Kieren, 1976, 1999; Steffe, 1998). That is, spoken symbols are used to describe fractional knowledge at a range of different levels of sophistication, allowing teachers and researchers to understand better and consider the relationship between children's use of fraction related language and their internal representations of the concepts at hand (Kieren, 1999). An example of this might be when a child describes receiving a *smaller half* of a chocolate bar than their friend; this use of language indicates the child is using the word 'half' to indicate a piece, or it may be that they are expecting it be equal and disputing it as a fair share.

Several studies have examined the relationship between the role of language and fraction performance (see Chow et al., 2016; Hansen et al., 2015; Seethaler et al., 2011; Vukovic et al., 2014); however, all used various measures to determine the relationship between their use of language and fraction performance, and they classified fraction performance in varying ways. For example, comparing fraction competency to whole number knowledge (e.g., Seethaler et al., 2011), performance of fraction procedures and concepts (e.g., Chow et al., 2016; Hansen et al., 2015) or numerical competency more broadly. The results are mixed as to the relationship between language (such as expressive vocabulary) and fraction performance; therefore, conclusions are hard to draw in terms of the relevance these studies provide for early years children.

According to Ball (1993), it is the role of the educator/researcher to interpret and understand the role spoken symbols/words/language play in the context of each learning opportunity. What this means for fraction instruction in the early years is that when children are engaging and participating in the act of partitioning, for example, educators are identifying and accurately interpreting the child's use of language to determine how they are conceptualising fair shares, in that context – whether that be folding paper or dealing out counters.

In the context of the present study, spoken symbols and language relate to the way in which children name the quantities they are exploring. For example, referring to the flowers and vases example in section 2.3.4, depending on how the child is describing and naming the share or relationship indicates a different fraction meaning, illustrated by an individual idea (such as many-to-one, 1-nth-of... or many-as-one as an example). The development of fraction language in the context of this study is, therefore, an important characteristic to identifying how children are conceiving such ideas between whole and fractional quantities. That is, they way children describe the quantities—in which the differences in their explanations may be very subtle—are key to identifying which meaning of fraction they may be developing. Moreover, given the spatial nature of the ideas and context that will be explored through these three meanings of fractions, the children's use of language in terms of the changes and manipulations they are physically and or mentally making to objects will be important to identifying the strategies for *how* they are partitioning, unitising, or considering quantitative equivalence as examples.

2.5.2.2 Real-World Contexts

Real-world situations refer to exploring a mathematical concept in a familiar context for the child, or at least a plausible situation (Lesh, 1987). It assumes that children can use their informal understandings in a learning context and successfully build on this knowledge when then concepts are presented in a real-life context (Brown, 2019; Leinhardt, 1988). In the early years, this involves using authentic contexts where fair sharing is embedded, such as distributing a fair share of craft materials per child or taking turns on the swing for an equal amount of time. Baroody and Hume (1991) argue that instruction should involve purposeful learning that links different forms of representation (both internal and external) to the various concepts and meanings of fractions. Utilising real-world contexts and connecting to other forms of representation appropriate to the age and informal knowledge of the children is said to mitigate some of the difficulties many learners experience with fractions (Baroody, 1989; Clements & Del Campo, 1987). A widely acknowledged theoretical framework that explores the notion of real-world contexts is Freudenthal's Realistic Mathematics Education (Streefland, 1993a). This is a domain-specific theory for mathematics education introduced as a response to traditional curricula approaches that perpetuate closed, isolated learning contexts to the notion that mathematics is considered as an activity of mathematisation. That is, learners use mathematics to organise and solve real-world contexts while reinventing and mathematising new understandings as they participate in the real-world context (Freudenthal, 1973; van den Heuvel-Panhuizen & Drijvers, 2020).

Examples of real-world contexts can include children connecting fraction ideas to their explorations of sand and water mixes during play or determining how to use fraction and proportional understanding to use a map and find different landmarks within a physical space. These real-world fraction and proportional contexts are deeply connected to children's typical play and early educational experiences, which draw on spatial reasoning skills and abilities as described above. The connection between fractions and spatial reasoning in the early years is often facilitated by the real-world contexts with which children intuitively engage; however, it is imperative that teachers and educators are aware of where early fraction ideas are implicit within a context. Ensuring that children have opportunities to build on their fair sharing and distribution ideas that they develop through play, cooking, craft making and sandpit play, as described in previous examples in this chapter, highlight how fractions are an important mathematical concept embedded in young children's worlds.

2.5.2.3 Concrete Manipulatives

Concrete or physical manipulatives are synonymous with early childhood education. From the influence of seminal works such as Montessori (1912), Piaget (1964) and Bruner (1973), objects and manipulatives have been touted as instructional tools that can provide opportunities for children to learn abstract concepts (Beilstein, 2019). For example, for the development of fraction understanding, concrete materials typically include paper-based models (both circular and rectangular), Cuisenaire rods, fraction tiles, pattern blocks or geometric shapes, plasticine and coloured counters that allow children to develop and represent their fractions ideas (Ojose & Sexton, 2009).

There are mixed results in the literature on the use of concrete materials to support the development of fraction ideas in primary school. Some studies revealed that children who engaged with manipulatives during fraction instruction had better retention or transfer in the post-measure of the intervention. For example, in fraction as a measure problems, Cramer and Wyberg (2009) found that a static fraction bar chart did not support Year 4 and 5 children to estimate the changes in quantity when fractions were added or subtracted, but it did support children's understanding of the magnitude of unit fractions when comparing denominators (see Cramer & Wyberg, 2009). Similarly, they found that children who had trouble manipulating pattern blocks when ordering fractional parts also demonstrated difficulty in 'the construction of mental images for fractions' (p. 255). However, the authors questioned how well the teachers implemented these models across the study, suggesting more research on these models for teaching and learning fractions is warranted. These findings suggest that the teacher needs to have a clear understanding of and intention for how the model is to be used and interpreted, otherwise concrete materials may not be entirely useful.

The common theme underpinning contemporary theoretical perspectives for teaching fractions is that effective instruction includes exposing children to the different meanings of fractions for conceptual development. However, research indicates that the key difficulties young children exhibit in developing early fraction ideas are concerned with making the connections between the different meanings of fractions and the concrete representations in which they are explored (Bobis & Way, 2018; Way et al., 2015). For example, when a typical representation of fractions, such as a circular model, is presented, the child needs to move beyond the perceptual attributes (such as colour and shape) to recognise what relationship the parts are representing in relation to the whole and so on. Ball (1993) argues that no concrete material and representation is a panacea and that although concrete materials and models are fundamental to mathematics itself (Mainali, 2021), they must be fit for purpose in their intention, relationship to the concepts and mathematical ideas and how they are perceived by children (Baroody, 1989).

Although there is much debate about the types and prevalence of concrete materials for teaching mathematics generally, the literature on concrete materials and fractions suggests it does not matter what concrete material is employed; what matters is that children develop meaning and a purpose for what they are learning with the assistance of the chosen material (McNeil & Uttal, 2009; Sarama & Clements, 2009). Moreover, in the present study, this means that the models and materials I select for the study's intervention must enable the children to develop a meaningful and conceptual understanding of the intended mathematical ideas.

2.5.2.4 Pictorial Representations

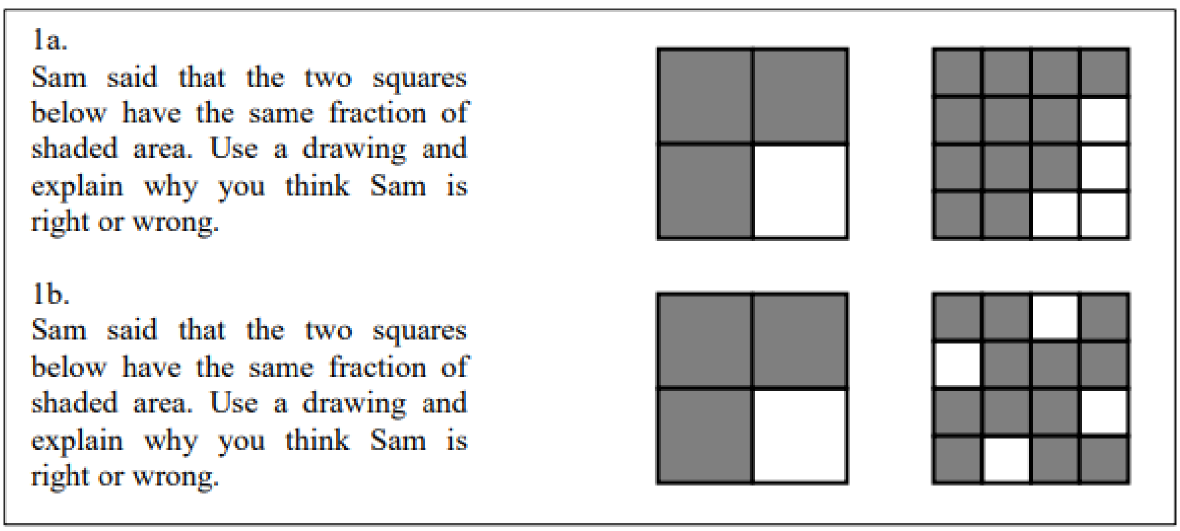
Pictorial representations (i.e., drawings, graphs, and diagrams) are described by Woleck (2001) as placeholders for thoughts, which can enable children to work on one part of a problem mentally, without experiencing cognitive overload (Anderson-Pence et al., 2014). They can be provided to children as visual aids and scaffolds or can be the result of children representing a problem, serving as a metaphor for mathematical understandings (Woleck, 2001). This reiterates the spatial reasoning component of pictorial representations described by Arcavi (2003), as his definition of mathematical visualisation entails the ability to create, utilise, interpret, and reflect on images both presented on paper and in the mind. This suggests that there is a deep relationship between how pictorial representations influence and are influenced by children's internal representations of mathematical ideas (Anderson-Pence et al., 2014; Tall & Vinner, 1981). For example, MacDonald (2013) found that when young children were asked to communicate their understanding of measurement, (specifically time and mass) through pictorial representations, the images provided a powerful tool for accessing how children created and communicated their understandings. That study also revealed that the pictures were not just representations of procedures by which children record their knowledge about a concept; the pictures revealed the processes through which understandings might be constructed, reconsidered, and applied in new ways throughout various contexts (MacDonald, 2013). These findings suggest that pictorial representation by young children connect to their real-life contexts in which they explore mathematical idea and is an important way for them to represent their mathematical ideas.

One of the most common difficulties children have with pictorial representations and developing fraction understanding is creating inappropriate or ineffective representations of the problem. For example, Misquitta (2011) found that the most common pictorial representations children in the early years of schooling use is the fraction circle, which emphasises the part-whole idea. This limited representation of fraction means children are not supported to think of fractions greater than one, which has negative implications for develop rational number ideas more broadly (Behr et al., 1983; Misquitta, 2011).

A second issue with the use of pictorial representation is a lack of understanding about the images used. For example, Anderson-Pence et al. (2014) explored Year 3 and 4 children's understanding of equivalence using two tasks. The first task required children to compare rectangular area models (see Figure 2.17).

Figure 2.17

Area Task for Year 3 Children



Note. From 'Relationships between visual static models and students' written solutions to fraction tasks,' by Anderson-Pence, K. L., Moyer-Packenham, P. S., Westenskow, A., Shumway, J., & Jordan, K. (2014). *International Journal for Mathematics Teaching and Learning, 15*, 1-18. Copyright 2014 by Katie L. Anderson-Pence, Patricia S. Moyer-Packenham, Arla Westenskow, Jessica Shumway, Kerry Jordan. Reprinted with permission.

The second task was a pizza problem that required children to draw a pictorial model to prove 'who ate more' (see Figure. 2.18).

Figure 2.18

Instructions for Pizza Tasks

Think carefully about the following question. Write a complete answer. You may use drawings, words, and numbers to explain your answer. Be sure to show all of your work.

a. José ate ½ of a pizza
Ella ate ½ of another pizza

José said that he ate more pizza than Ella, but Ella said they both ate the same amount. Use words and pictures to show that José could be right.

b. A pizza is sliced into 10 equal pieces and José ate 4 slices of the pizza.
Another pizza is sliced into 5 equal pieces and Ella ate 2 slices of the pizza.

José said that he ate more pizza than Ella, but Ella said they both ate the same amount. Use words and pictures to show that José could be right.

Note. From 'Relationships between visual static models and students' written solutions to fraction tasks,' by Anderson-Pence, K. L., Moyer-Packenham, P. S., Westenskow, A., Shumway, J., & Jordan, K. (2014). *International Journal for Mathematics Teaching and Learning*, *15*, 1-18. Copyright 2014 by Katie L. Anderson-Pence, Patricia S. Moyer-Packenham, Arla Westenskow, Jessica Shumway, Kerry Jordan. Reprinted with permission.

Anderson-Pence et al.'s (2014) findings revealed that children exhibited difficulties when interpreting pictorial representations, such as whole number interference, and a limited understanding of fractions (Anderson-Pence et al., 2014). That is, they counted the number of squares coloured rather than considering the squares as a proportion of the whole (e.g., they might say '12' rather than '3-quarters'). They also assumed that the two pizzas in the second task were the same size, meaning they were generally unable to provide accurate representations of the solutions. This means children need to be exposed to a rage of pictorial representations that enable them to build connections between the fraction symbols and meanings.

That study was conducted using multimodal forms of pictorial representations, including virtual, physical, and static forms. Thus, the results even though the children had opportunities to manipulate and explore the different forms of pictorial representations of fractions, there was a large gap between the children's ability to generate and use the pictorial representations effectively to solve fraction problems. For children in the early years of schooling, this is an important point as it is unlikely that children will have experiences with the extended range of fraction ideas explored in the study; therefore, they will likely have limited understandings of how they can accurately represent such context. Enabling children to develop a connection between concrete models and their own representations is key to enabling children's development of the three meanings of fractions.

Pictorial representations are well embedded within early childhood mathematics, and the area of fractions is no exception, yet we cannot assume that the graphics we present to children or that they create are interpreted as intended. As Lowrie (2012) and others have established (see Lowrie & Logan, 2007; Diezmann & Lowrie, 2008; Lowrie & Diezmann, 2007; Logan & Greenlees, 2008), children's spatial reasoning abilities need to be closely developed within the context of creating and interpreting pictorial and graphic representations when solving mathematics tasks. For example, spatial ideas such as perspective taking, location and orientation, size, and scale of objects within 2D representations need to be explicitly considered and understood when interpreting the fraction problem represented. An example of such a task is Möhring et al.'s (2016) cherry juice task, previously depicted in Figure 2.13. Here, consideration of the relative size of each of the regions needs to be considered in relation to their mixtures to determine, proportionally, which is sweeter. Teachers need to carefully help children determine what the image represents so that children can make sense of the underlying concepts (in this case, relating to fractions) and understand how and why pictorial representations may or may not be appropriate across different contexts.

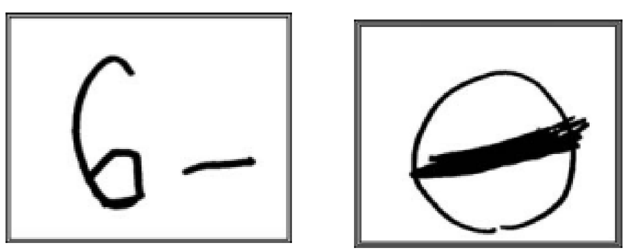
2.5.2.5 Written Symbols

Written symbols are an essential component to communicating mathematical ideas and relationships, and fractions are no exception. Young children begin experimenting with written symbols through creating idiosyncratic marks and informal notations that are used to communicate young children's thinking about mathematical meanings (Worthington & Carruthers, 2003); for example, making marks to represent a quantity, or attempting to draw a 2D representation of a place or space like their house.

There is an emphasis in early mathematics teaching on children representing formal, numerical symbols for whole numbers, namely in matching names and numerals to collections and the early development of counting (Baroody, 2001; Munn, 1998). Nevertheless, many researchers have stated the emphasis on whole number notation can interfere with children's conceptualisation of fractions, which feeds into the part-whole view and promotes the WNB, and double counting phenomenon discussed earlier in this chapter (Gould, 2011; Mix et al., 1999; Ni & Zhou, 2005; Saenz-Ludlow, 1994; Sophian et al., 1997). Regarding fraction symbols and their introduction, Kieren, cited in Huinker (2002) states that premature experiences with formal symbols and their procedures (e.g., algorithms) may lead to symbolic knowledge that is not based on a deep conceptual understanding, or connected to the real world. Without deep conceptual and pedagogical understanding, the emphasis on written symbols can often come at the expense of using concrete and or pictorial models, which Bruce et al. (2013) state, 'has the potential to impede students in developing fluency across the different representations of fractions' (p. 22). However, Brizuela (2006) extends Empson's (1999) ideas that the use of written symbols performs a transformative function between the development of conceptual understanding of fractions and various representations (such as concrete models, pictorial representations, etc.) in which young children typically engage. In Brizuela's (2006) study, 5 – 6-year-old children exhibited fraction ideas through written symbols (called *notations* to infer both formal and informal written symbols) in conjunction with other tools of representations such as pictorial and concrete materials. Children were interviewed after receiving a traditional mathematics curriculum that was guided by textbook instruction. This instruction included some formal symbolic notation of common fractions (for 6-year-old children only). The clinical interviews were conducted one on one, where children were asked a series of questions and to represent their answers (such as what they knew about half, how they would represent their age, how they would represent a fair share outcome). Although the majority of the children in that study *did not* use formal, symbolic notations within their fraction work, the way in which they recorded fractions revealed that children utilise and engage with informal written notations to conceptualise fraction magnitude idea, such as that represented in Figure 2.19.

Figure 2.19

Zachary's Representation of His Age: 6 and - Half



Note. From "Young Children's Notations For Fractions," by Brizuela, B. M, 2006, *Educational Studies in Mathematics, 62,* 281–305. Copyright 2006 by Springer Science Business Media, Inc. Reprinted with permission.

Zachary's written description of his age started with a '6' and a line to represent his age of 6-and-1-half years. When asked what the line meant in terms of his age, he then drew a circle and emphasised the line through the middle as half, to represent this part of his age. Although informal, these symbols provide valuable insights into children's thinking and understanding of fraction magnitude. This shows it is important to provide opportunities for recording their ideas and understanding of fractions in ways that help them to reveal their understandings.

Similar findings have also been highlighted in other studies involving children of similar ages and their informal written symbols for communicating ideas about fraction quantity and magnitude (e.g., Ball, 1993; Empson, 1999; Mack, 1990; Pothier & Sawada, 1983; Saenz-Ludlow, 1994, 1995). Therefore, encouraging children's emergent written symbols for developing the range of fraction concepts and interpretations 'provide windows to develop more nuanced and complete pictures of their ideas about fractions' (Brizuela, 2006, p. 301) in which to build on.

The overall learning of fractions cannot be disassociated from the words and numbers used to represent them (Mamede et al., 2005). Yet, as many of these children did not have the understanding and knowledge to represent fractions symbolically in this study, the introduction of formal symbolic notation early in the development of these ideas may be unnecessary. Saenz-Ludlow (1994) argues that formal symbolic notation should be delayed until students have conceptualised fractions as quantities through estimation and experimentation (see also Pearn & Stephens, 2007). However, this does not mean informal and idiosyncratic symbols are not an important immediate step in the development of formal written symbols. An intermediate step in the development of the formal written symbols suggested by Siemon (2013) is to use a combination of written words and symbols in the format of, for example, *3 fifths*, *2 halves*, *3 sixths*, to distinguish between the count (how many parts) and the value or size of the part (how much each is worth). This focus on recording early fraction quantities in this way emphasises the distribution of one composite unit over another, which is an important step in developing the multiplicative foundations of fractions (Siemon, 2013).

Given the evidence above that some children can engage with symbolic notations in the representations of magnitude for fractions, the literature suggests that the transition to formal symbolic notation needs to develop gradually (Boyer & Levine, 2012; Siegler & Lortie-Forgues, 2014) and be supported by the way in which we use fraction language. This is an important consideration for children in the early years because it assists them to think about the fraction as a quantity in its own right, helping to establish an understanding of the connection between fraction magnitude and symbolic representations.

2.5.3 Examining Representations Beyond the Translation Model

Throughout the review of literature pertaining to external representations and young children's mathematics, it became apparent there is a body of work that suggests gesture is an external representation that can mediate mathematical meaning in early childhood contexts, particularly in relation to learning and communicating spatial information (Alibali et al., 2014; Bobis & Way, 2018; Edwards & Robutti, 2014; Elia & Evangelou, 2014). This is not represented in Lesh et al.'s (1987) translation model; however, the prevalence of gesture within some of the literature on other representational forms (such as the role of language and manipulatives) in the present review suggests it is worthy of exploration.

2.5.3.1 Gesture

Gesture as an external representation has become more of a focus within mathematics education over recent years (Gerofsky, 2014). Alibali (2005) argues that gesture plays a significant role in communication and in the cognitive processing of spatial information; thus, a gesture is considered a tool and form of representation. The visuospatial nature of gesture makes it suitable for capturing spatial information because it brings imagined or abstract spaces and objects into a more concrete form. Further, gestures represent spatial properties and action-based characteristics of concepts (Krauss et al., 2000) by assisting speakers to activate mental images and maintain these spatial representations in working memory (Alibali, 2005). For example, several studies have revealed that people use more gestures when asked to describe an object or image when the object or image was no longer visible (such as a shape or a still-life painting where the images and objects were not easy to describe verbally (see De Ruiter, 1998; Moresella & Krauss, 2004). At the same time, using gestures to express spatial properties can help activate related mental representations of the concepts in verbal form, thus linking one internal representation to the external form (Krauss et al., 2000). In addition, there is evidence to suggest that producing gestures facilitate speakers exploring possible ways of packaging spatial information into a 'verbalisable unit, by exploring alternative ways of creating and organising spatial and perceptual information' (Alibali et al., 2000, p. 595). Thus, gesture can be considered a way of supporting learners to better remember concepts and ideas.

McNeill's (1992) conception of gesture is a useful way of thinking about this form of embodied cognition as a theoretical grounding for this discussion. For McNeill, a gesture is understood as hand movements that represent meaning in relation to accompanying speech, called gesticulations. Gesticulations (hereafter referred to as 'gestures') are categorised by McNeill using the following categories: *iconic*, *metaphoric*, *deictic,* and *beat*.

A gesture is classified iconic if it bears a close relationship to the content of the accompanying speech, such as raising one hand slowly when stating, 'I walked up the hill'. Here, the gesture is closely connected to the movement described in the speech, complementing the description to provide a sense of context for the recipient of the communication. In the context of fractions, an iconic gesture may be hand movements that represent the sharing or 'dealing out' of a set of items one by one (such as sharing 12 lollies between three children), indicating the process of partitioning in a fraction as quotient context.

A metaphoric gesture is similar to an iconic gesture, with the difference being that the gesture represents an abstract idea that cannot be physically seen, such as knowledge, language itself, the genre of a narrative (McNeill, 1992). Metaphoric gestures are represented in two parts: the base, or physical action represented in the gesture, and the referent or concept that is the meaning represented by the base. An example of a metaphoric gesture is ‘blowing a kiss’. The action of touching one’s lips with one hand and moving them away while pursing one’s lips together is the base of the gesture. The referent is the concept of giving love or affection to the recipient of the gesture.

Deictic gestures are typically represented as a pointing movement using the hand’s index finger, but can also be used with the head, nose, eyes, or chin, depending on the sociocultural context and norms of the situation to point to a particular object or space during speech.

Finally, beat gestures do not necessarily present a relationship to the content of the speech; instead, they may indicate an emphasis on the rhythm and flow of the accompanying speech. For example, someone may tap their finger at the same time they state a particular word or phrase for emphasise.

In terms of studies that examine children’s use of gestures when learning fractions, Mildenhall (2013) and Takeuchi and Dadkhahfad (2019) found that Grade 6 and Grade 4 students, respectively, used iconic gestures when communicating their ideas about equivalent fractions, which gave insight into the students’ understanding that was not apparent through traditional pen-and-paper representations. Beilstein (2019) analysed video recordings of 26 children from Grades 2 to 5 in a mixture of mainstream and gifted education classes. The children were not provided with activities, pictures, or manipulatives with which to work; they were simply questioned on their knowledge of fractions. The findings indicated that 96.2% of children gestured a whole object (primarily a circle or rectangle); 92.3% gestured a cutting, chopping and or pinching action to represent parts of a whole; 42.3% gestured symbolic notation; and 11.5% indicated a number line through gesture, by sweeping their hands along a vertical or horizontal plane, and pointing to specific points along that line to describe magnitude. These data illustrate that iconic gestures were common across all age groups and classes. While that study did not explore the potential influence of a pedagogical emphasis of gesture on children's fraction understanding, it does give insight into the way gesture is utilised as a representation.

Swart et al. (2014) reconceptualised the role and description of gestures based on McNeill's (1992) taxonomy to enable this form of embodied cognition to be assessed using digital tablets during a series of fraction tasks. Their studies produced two categories of gesture. First, *conceptual gestures*, which included metaphoric and iconic gestures. These were typically in the form of children drawing a symbol in the air or drew the procedure or algorithm as they explained. Second, *deictic gestures*, which referred to on-screen pointing or swiping across the screen. Across Years 3, 4 and 5, students who performed better at partitioning and estimating the magnitude of unit and composite fractions used far more conceptual gestures than dietic gestures. For example, children demonstrated the process of partitioning by using a slicing action to indicate a fractional measure, utilising 'gesture as simulated actions' (Hosetter & Alibali, 2008, p. 502). This provides some insight into how children might think about the operation of partitioning and how they create different quantities, which is an important consideration for working with children in the early years as they typically have limited experience with a range of extended fraction ideas.

From a pedagogical perspective, Edwards (2008, 2009) explored the gestures used by elementary PSTs when interviewed about how they were first introduced to fractions as children, the difficulties they experienced learning, and how they use fractions in their everyday lives and other university classes. They were also asked to solve a range of written fraction problems and asked at the conclusion of the interview to explain how they define a fraction and how they would introduce this topic to children.

Using McNeill's (1992) scheme as a coding tool, Edwards then coded the gestures directly connected to talk about fractions, which revealed 40% of gestures identified were metaphoric and 35% were iconic. The remaining 25% were a combination of beat and deictic gestures that were not explicitly discussed in these studies. Metaphoric gestures were most prevalent when the participant described an abstract object or idea (such as the value of a fraction in relation to an imagined object) and used their hands to represent the magnitude of the fraction concerned. The most prevalent iconic gestures pertained to the concept of partitioning, which saw participants using their hands to represent a cutting, sawing, or splitting action. They also included drawing algorithms in the air described by Edwards (2008) when participants referred to symbolic notation and the processes involved in operating with such algorithms. Surahmi et al. (2018) found similar results in the way the types of gesture occurred in a study involving Grade 3 teachers. However, iconic gestures of drawing algorithms in the air seemed to be more prevalent in this context, possibly because of the context of the Indonesian education system, which is heavily based on didactical pedagogies (Sembiring, 2008).

Considered together, these studies suggest that gesture may be a way children explicate and communicate various fraction ideas. How young children may engage spontaneously with gestures in addition to other concrete or abstract representations such as those above does not appear to be researched or investigated explicitly in the early years of primary school. Not unexpectedly, all of the studies considered here acknowledge that the affordances and synergies between gesture and mathematics need further exploration, something that will be explored in the present study. Although the limited research on gesture and fraction learning has been associated with adults or children older than those of interest in the present study, this literature is beneficial in understanding how children may use gestures to communicate their understandings about fractions.

2.5.4 Spatial Reasoning and Fractions Summary

A review of research on the influence various aspects of spatial reasoning has on young children's fraction development revealed clear connections between these two critical areas of cognitive development. While spatial reasoning is not a single skill or ability, specific constructs can be described that help untangle the complexity of this transdisciplinary way of thinking. Literature about young children's fraction development suggests spatial proportional reasoning and spatial visualisation are key spatial constructs associated with the development of internal representations of fraction ideas and concepts.

Further, external representations are an essential component of early mathematical experiences. They are connected to spatial reasoning, specifically in how images and materials may be perceived from their spatial attributes and used to convey information about relative magnitude and quantity. Gesture may act as an intuitive tool for communicating internal representations on fraction ideas. There is little research on the efficacy of this form of representation for young children during sustained classroom instruction.

This discussion illustrates the relationship and interaction between internal and external representations are often mediated by spatial reasoning constructs for younger children. That is, spatial proportional reasoning is often used to describe the quantities in proportional contexts (see Frick & Newcombe, 2012; Huttenlocher et al., 1999; Jirout & Newcombe; Möhring et al., 2014; Vasilyeva & Huttenlocher, 2004). Additionally, there is research that suggests children's conceptualisation of magnitude can be supported by visualising what the outcome may be when imagined actions are performed on an object (such as engaging in partitioning, unitising, and quantitative equivalence) in both two- and three-dimensional spaces (Clements & Battista, 1992; Lamon, 1996; Mix & Levine, 2018). This review suggests there is a justification for a more explicit focus on the relationship between spatial reasoning constructs and external representations that are embedded in the development of early fraction knowledge and how these might develop in naturalistic, intervention-based settings.

2.6 Central Insights Framing This Study

The present study is concerned with examining how spatial reasoning may assist in young children's development of a range of fraction meanings. Based on the analysis of various theoretical and pedagogical perspectives, the following insights have been derived from the literature, which will be used in the design of this study.

First, it appears the underpinning starting point children need to establish to work with an extended range of fraction ideas is an understanding that a continuous object or set can be divided into equal parts or fair shares. That is, Confrey's (1994) perspective on splitting suggests that a partitioning approach offers a multiplicative foundation to exploring early fraction concepts (highlighted by their Equipartitioning learning trajectory within their rational number framework). This framework emphasises and illustrates how the partitioning approach involves the exploration of fraction as ratio, fraction as an operator, and fraction as a measure meanings simultaneously, which suggests it supports children to establish an understanding of fractions more authentically than a traditional approach offers.

Second, children need to develop strategies that help them to identify and name quantities from different perspectives. That is, for children to develop flexible ideas about fractions, they need to view quantities from different perspectives (e.g., *if six lollies are half of the bag*, *one fair* *share of the set is six, two shares make a whole, for each person there are six lollies*). This draws on children's visualisation processes for considering the relationship between the quantities generated and the whole. Visualising the outcome of creating different shares within the same whole by exploring spatial proportions of like wholes is foundational to this insight and for the development of a sense of magnitude of various quantities.

Third, it appears that several theoretical perspectives (e.g., Confrey, 2008; Matthews& Ziols, 2019; Möhring et al., 2016; Siegler et al., 2011) suggest that young children are sensitive to spatial ratios, visually comparing parts and regions and predicting the outcome of creating equal shares in both continuous and discrete contexts. This suggest that early proportional and fraction as a relation ideas are accessibly to children through spatial reasoning contexts.

Fourth, the review of representations and early fraction contexts suggested that children perform acts of partitioning, by doubling and halving and redistributing various materials in a concrete form during their play activities. Therefore, mentally manipulating continuous wholes such as splitting, reassembling, and doubling and halving simple ratio units to identify these relationships as examples suggests that suggest they could be promoted by spatial visualisation and spatial proportional reasoning abilities to make connections between a wide range of early rational number ideas.

To conclude, this review has explored the debate about the development of fractions in relation to whole number ideas from a range of theoretical perspectives. The present study takes the perspective of Confrey et al. (2014b) in that partitioning and early fraction ideas develop separately or in parallel to children's whole number and counting abilities because of the hypothesised role spatial reasoning plays in this development. Moreover, using an approach based on Confrey's framework is hypothesised to mitigate a counting and part-whole overreliance in children's early fraction experiences that is frequently reported in the literature.

Based on this perspective, the present study suggests that children may develop early multiplicative relations of numerical quantity that involve both fraction and whole number relations, if provided with the opportunity to do so.

These four central insights derived from the literature enabled two research questions to be developed.

2.6.1 Research Questions

Research Question One: To what extent and in what ways can young children demonstrate an understanding of an extended range of fraction ideas experienced through a spatial reasoning approach?

Research Question Two: To what extent, if any, does this approach to fractions impact young children's understanding of whole number?

2.7 Chapter Summary

This chapter has identified the prominent theories of fraction knowledge that have underpinned educational research over the past 50 years. The review applied a transdisciplinary lens that revealed other cognitive factors such as spatial reasoning and different forms of representations are influential in the engagement with the development of the concepts (partitioning, unitising, and equivalence) and the various meanings of fractions (part-whole, measure, quotient, operator, and ratio) for young children.

This study describes several theoretical foundations of how fractions are conceptualised, which typically form two main pedagogical approaches: the measurement, and partitioning (splitting) perspectives. The measurement perspective (supported by Steffe & Olive [2010], Siegler et al. [2014] and Tzur [2019]) focuses on a measurement (as discrete counts) interpretation for fractions, which, as described above, can limit the opportunities for children to understand the complexity of this area of mathematics (Clarke, 2011; Lamon, 2007). Conversely, the partitioning approach is founded on multiplication and division structures needed to conceptualise operator, number, and ratio meanings (Bruce et al., 2014; Confrey & Maloney, 2015). Also evident is that an understanding of partitioning is dependent on the development of multiplicative foundations of rational numbers, not counting or whole number contexts (Bruce et al., 2014; Confrey, 2008; Pepper, 1991). Confrey et al.'s (2014b) rational number framework provides a basis for which an intervention can be developed and explored with young children. However, the evidence of children engaging with various meaning of fractions suggests young children do so through the explicit use of spatial reasoning skills and strategies.

Synthesised from the literature on spatial reasoning and its relationship to fractions, an initial conjecture for the present study was that specific spatial constructs of spatial proportional reasoning and spatial visualisation are beneficial for children to engage with when learning early rational number ideas. Further, young children's capabilities with early rational number ideas have been examined and found to be associated with spatial reasoning contracts, as evidenced in a number of experimental studies. However, the connection between how spatial reasoning may be promoted in a pedagogical sense within early primary school has not yet been fully explored. Moreover, the outcomes for children's representations and understanding of early fraction ideas is also under-researched, suggesting that a spatial reasoning approach in the context of a classroom-based fraction intervention is a novel approach that may mitigate many of the difficulties children face in the development of rational number reasoning. Importantly, this review has revealed that spatial reasoning provides a vehicle for exploring such ideas and concepts.

Chapter 3: Methodological Considerations

3.1 Chapter Overview

This chapter begins by describing the theoretical perspectives that guided this study (section 3.2) and how it informed the choice of Design Based Research as the guiding methodology (section 3.3). The three overarching phases of the study are described in section 3.4. Section 3.5 describes the demographics of the participating classes and the characteristics of each school. Data sources are described and justified in section 3.6. Section 3.7 describes the data analysis processes, and a discussion on trustworthiness follows in section 3.8. Ethical considerations are addressed in section 3.9, and section 3.10 summarises and concludes the chapter.

3.2 Theoretical Perspectives

As presented in Chapter Two, a range of theoretical perspectives describe how children may construct fraction meanings. These perspectives describe how and when children engage in different fraction ideas and how spatial reasoning may support this learning. Based on this review, an interpretivist paradigm underpins the present study, as the foundation to this worldview is to understand the subjective world of human experience (Smith, 1992; Kivunja & Kuyini, 2017). That is, emphasis is placed on making meaningful conjectures about children's fraction development through examining how they experience novel learning approaches, with the recognition that the social world cannot be understood from any one individual perspective; rather, there needs to be a search for patterns in behaviour that can be observed within similar circumstances (Guba, 1981; Lincoln & Guba, 1985).

There is an assumption that the research problem and context being studied has multiple realities, which is reflected in the various theoretical perspectives presented in the literature review. These realities are explored, redefined, and contextualised through the experiences between the researcher and participants (Kivunja & Kuyini, 2017).

An interpretivist paradigm does not provide the researcher with descriptions of what to see; instead, this theoretical lens ‘merely suggests directions along which to look’ (Blumer, 1954, p. 7). Building on the transdisciplinary perspectives, the case was made for examining how children may develop an understanding of various fraction meanings within typical classroom environments through an approach they would otherwise not experience in typical instruction. Specifically, this research was concerned with:

1. examining the extent to which young children can develop fraction as operator, fraction as a measure, and fraction as a relation ideas through a spatial reasoning approach
2. analysing how young children engage in spatial reasoning and representations while exploring these fraction ideas
3. exploring and refining a local instruction theory about the learning of fractions in the early years.

As an inquirer, my role as the researcher is to elucidate the meaning-making process of the participants of this study by interpreting, analysing, and hypothesising how children engage in the learning environment (Shwandt, 1998) and to make sense of this for both theoretical and practical contributions to the field of education.

3.2.1 Epistemology

Young children construct and develop their understanding of early fraction ideas based on their prior knowledge and experiences within formal, informal, and social learning environments.

A blended theoretical perspective between the constructivist (Piaget, 1936; von Glasersfeld & Steffe, 1991) and sociocultural (Vygotsky, 1978) approaches to learning are the orienting theories that underpin this project (Prediger et al., 2015).

Constructivism is a theoretical perspective that suggests knowledge stems directly from the child, constructed and developed as the learner experiences the world. The sociocultural theory acknowledges such intrinsic development; however, it situates the construction of knowledge as a social and cultural practice, whereby the interactions with others deeply influence the child's potential for learning. There is consensus that these two perspectives are complementary because they are essentially rooted in the 'activity of attempting to understand what might be going on in a range of specific teaching and learning situations' (Cobb & Yakel, 1996, p. 175). Moreover, the complementary perspective provides a lens through which individual children's mathematical activity, the classroom environment and the broader pedagogical practices are considered to provide a comprehensive characterisation of the learning phenomena.

The reason for taking a complementary stance is because the tensions between these perspectives are endemic to the act of teaching itself rather than confined to theoretical contribution (Cobb, 1994). In other words, the integration of both learning theories allowed for a focus on the most critical aspects of this research, which is not that the children will produce correct solutions to problems involving fraction ideas in a controlled or artificially created social environment. Instead, it is concerned with whether children can produce insightful, meaningful, and flexible solutions when working with an extended range of fraction ideas through a spatial reasoning approach. Determining such behaviours and understanding involves examining both the individual's pathway to success and the effects of social context.

In this project, constructivist and sociocultural perspectives help to understand how learners access their prior knowledge experiences to create new ideas and understandings about whole number and fraction ideas. These understandings are scaffolded by using specific spatial reasoning strategies, which will be influenced by the learner's schemas and their collaborative experiences within the social environment. However, there are also no 'correct' or 'incorrect' theories about knowledge construction; rather, there are theories that seek to explain phenomena of interest (Walsham, 1995). Given the need to understand and reconcile the theoretical viewpoints about children's spatial reasoning and rational number development from the fields of psychology, neuroscience, and mathematics education, as described in Chapter Two, a complementary interpretivist stance allows me to explain children's individualistic mathematical behaviours while considering the mathematical development as it occurs in the social context of the classroom (Cobb & Yakel, 1996; Gravemeijer, 1998).

3.3 Design-Based Research

Design Based Research (DBR) is a methodology important to the mathematics education research community because there is a dual focus on (a) designing innovative forms of instruction to explore children's processes of learning and (b) refining local instruction theories for wider application and refinement (Confrey & Lachance, 2000; Prediger et al., 2015).

DBR seeks to explain the underlying meanings within the learning environment, consistent with the interpretivist assumption that multiple realities exist and are time and context dependent (De Villiers, 2005). Further, DBR supports the adoption of a complementary constructivist–sociocultural foundation because cognitive and conceptual development actions are fundamentally inseparable from the relations and social relationships between the researcher, participants, and the environment of the inquiry (Goldkuhl, 2012; Orlikowski & Baroudi, 1991).

Prediger et al. (2015) emphasise that DBR does not stipulate a strict set of methods, resulting in great variation in how DBR studies are designed and implemented. However, they describe five common characteristics that are a reference point for DBR researchers. These characteristics insist that a DBR study be:

- interventionist
- theory generative
- prospective and reflective
- iterative
- ecologically valid and practice oriented (Prediger et al., 2015, p. 879).

The *interventionist* notion of DBR provides an innovative context for learning that children would not otherwise experience in their typical instruction. The *theory generative* aspect of DBR is concerned with focusing on the processes of learning in these innovative contexts and generating a local instruction theory based on examining where the innovation takes children's thinking, what supports this thinking, and what this means more broadly for their subsequent learning. The connection between examining children's thinking and generating theory requires designing materials that provide the *prospect* for generating new insights into children's thinking and *reflectively* analysing children's engagement for redevelopment of the materials and revision of theoretical position. The connection between design, implementation and analysis requires an *iterative* approach where the intervention is examined and refined in various contexts, such as multiple classroom environments. Finally, the *ecologically valid* and *practice-orientated* foundations mean the findings and contributions of the research must be transferable into other contexts and inform instructional, research and design practice (Anderson & Shattuck, 2012;

Barab & Squire, 2004; Brown, 1992). The next section describes the research design of the present study, which reflects these central tenants of DBR.

3.4 Research Design

As stated in Chapter Two, the research questions guiding this study are:

Research Question One: To what extent and in what ways can young children demonstrate an understanding of an extended range of fraction ideas experienced through a spatial reasoning approach?

Research Question Two: To what extent, if any, does this approach to fractions impact young children's understanding of whole number?

These research questions are examined through exploring and refining a local instruction theory (Gravemeijer & Van Eerde, 2009) for fractions and spatial reasoning. Local instruction theories are informed frameworks for guiding the teaching and learning of a specific area of mathematics. They are considered similar to the notion of learning trajectories, in that they represent the learning processes that evolve in the development of a specific topic, and theories about the means of supporting such learning (Cobb et al., 2003; Gravemeijer, 1998; Prediger et al., 2015). However, as a reflexive tenant of DBR, the construction and exploration of local instruction theories serve as a basis for developing a domain specific theory, which serve as more global generalisations of multiple local instruction theories (Gravemejoer, 1998). In turn, the development of domain specific theories informs the exploration of additional local instruction theories as part of the DBR reflexive process (Gravemeijer, 1998).

The present study considers Confrey's et al.'s (2014b) learning trajectories for rational number reasoning framework as a domain specific theory for rational number reasoning. This view is on the basis that the framework is a connected network of multiple learning trajectories or local instruction theories, that provides the foundation for proposing an innovative local instruction theory for the three meanings of fractions specifically in the early years, that explicitly explores how spatial reasoning supports such learning.

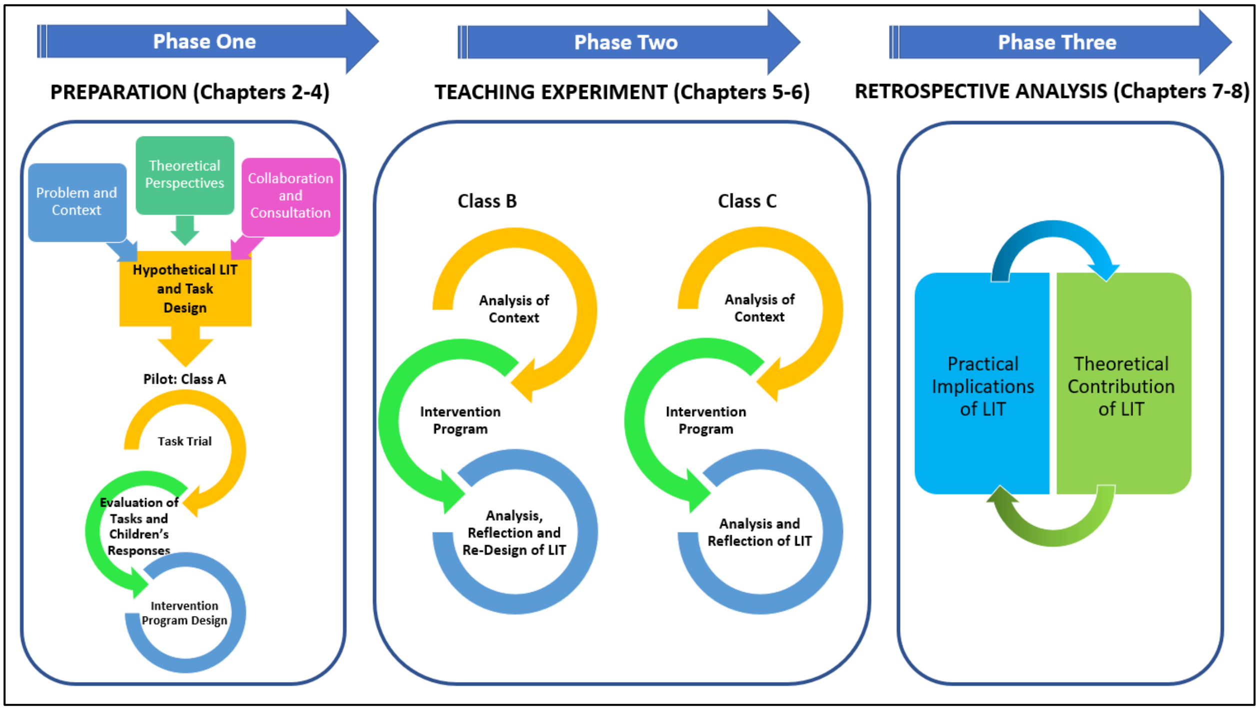
Further to this point, the connection between the ideas presented in Confrey et al. (2014b) framework relating to the early years have not been explored extensively, or in relation to spatial reasoning. In addition, the construction and refinement of a local instruction theory is not merely the adoption of current educational goals or ideals; it involves the problematisation of the topic to develop a paradigm case (also referred to as a 'humble theory'; Prediger et al., 2015, p. 885) to inform practitioners and researchers (Gravemeijer & Prediger, 2019; Prediger et al., 2015). In the present study, the selection of specific ideas within each of the three meanings of fractions was problematised in light of transdisciplinary research on children's potential to construct this knowledge and the role of spatial reasoning was conjectured to play in this development.

Local instruction theories are composed of three parts: 1) a series of learning goals that are conjectured on the basis of literature 2) a series of planned instruction activities and 3) rationale and evidence for how the activities support learning in a classroom setting (Gravemeijer, 2004). Regarding the description of learning goals, I have chosen to name these as *key indicators*. Given the innovative approach this study is taking, the fraction ideas and spatial reasoning emphasis is likely to be unfamiliar to the children. Therefore, I expect that the children will develop and demonstrate *indications* of these goals, but they may not fully establish each goal in the given timeframe.

To explore the conjectured local instruction theory, this DBR study comprised three overarching phases: 1) the preparation phase, 2) the teaching experiment and 3) retrospective analysis (see Figure 3.1).

Figure 3.1

Representation of the Present Study



Each phase is now explained in the following sections.

3.4.1 Phase One: Preparation

The preparation phase consisted of four main components:

- construction of a conjectured local instruction theory proposed on the basis of the literature review
- development of a range of tasks to promote each key indicator in the local instruction theory
- trialling the suitability of the tasks for inclusion in the intervention program and examining the range of children's responses and strategies generated by the tasks
- revision of the key indicators of the local instruction theory and sequencing of the intervention program as a result of this trial.

The local instruction theory was proposed by examining what is known about young children's fraction competencies and making conjectures about how spatial reasoning may support this learning. Given that 'design-based research places much value on the input of practitioners and researchers working in, or investigating, the problem area' (Herrington et al., 2007, p. 5), this preparation period was not undertaken in isolation. Several stakeholders were consulted during this phase of constructing the conjectured local instruction theory. The collaborative process involved discussing my understanding of how children develop various fraction ideas based on the literature with my PhD supervisors. In addition, five Foundation (first year of school in Australia) to Year 2 classroom teachers who had expressed interest in participating in this study discussed their approaches to teaching fractions. This process enabled me to ensure I was developing innovative tasks and activities for the children that they would not have experienced otherwise, in consideration against the literature.

In Chapter Two (section 2.6), four central insights were derived from the review of the literature and used as a basis to develop the series of key indicators. The first insight was that the overarching problem children exhibit in the working with fraction is that they are not provided with opportunities to adequately develop the foundations of partitioning—that is, creating equal shares—in both continuous and discrete contexts. The idea of a fair share is a critical starting point for the development of fraction understanding; however, in the early years, the literature suggests fair sharing and equal parts of discrete and continuous collections develop from children's spatial awareness of the materials and models used. That is, children need to visually compare, manipulate, and then visualise this operation establish the idea of equality and fair sharing. Based on these insights, the first two key indicators were proposed (see Table 3.1).

Table 3.1

Conjectured Key Indicators One and Two of the Local Instruction Theory

Key Indicators	**Characteristics of tasks** **Proposed Fraction Foci**	 **Proposed Spatial Reasoning Foci**	**Supporting Literature**
Establishing equal parts of collections of discrete items	**Fraction as an Operator:** Fair share Doubling and halving Partitive division/ recursive multiplication **Fraction as a Measure:** Many-as-one	Visual perception of equal groups (drawing on subitising). Recognising relationship between creating shares and recreating the whole from its parts.	Confrey et al. (2014b); Matthews and Ziols (2019); NRC (2006)
Establishing equal parts of continuous items	**Fraction as an Operator:** Fair share Doubling/halving Partitive division/ recursive multiplication Equi-partitioning a single whole Geometric symmetries Similarity **Fraction as a Measure:** Many-as-one	A focus on concepts of space for geometric parts—shape, orientation, symmetry in continuous wholes. Visualising the relationship between the shape and size of parts created in relation to the whole.	Bruce et al. (2015); Confrey et al. (2014b); Möhring et al. (2015)

The second insight into children's potential to work with an extended range of fraction ideas was that children needed to explore how a range of fair shares can be created in relation to their whole and how this process develops. That is, they need to visually compare different parts and how they are created with regard to the identified whole and explore naming the quantities in different ways that reflect the three meanings of fractions. Based on this, the next two key indicators were developed (see Table 3.2).

Table 3.2

Conjectured Key Indicators Three and Four of the Local Instruction Theory

Key indicators	Characteristics of tasks: Proposed Fraction Foci	Characteristics of tasks: Proposed Spatial Reasoning Foci	Supporting Literature
Reinitialising the unit	**Fraction as a Measure:** Unit and composite fraction Equivalent fractions **Fraction as an Operator:** Doubling and halving Times-as-many 1-nth-of... …	Visualising measures between parts and wholes, composite and unit fractions, equivalent units. Visualising magnitude relations between parts the distribution of parts.	Confrey et al. (2014b); Bruce et al. (2013); Clements and Sarama (2014; 2017/2019); Confrey and Smith (1995); Siemon et al. (2017)
Splitting as a mental act	**Fraction as an Operator:** Partitive division/ recursive multiplication Times-as-many 1-nth-of...… **Fraction as a Relation:** Many-to-one Distribution	Visualising the relationship of partitive division/recursive multiplication, times as many. Stretching/shrinking geometric wholes.	Behr et al. (1983); Confrey et al. (2014b); Lamon (1999)

Finally, for children to develop an extended range of fraction ideas, it was determined that they needed to understand the relationships between fraction and ratios by developing ideas about unit and proportional equivalence. Although it is not expected that young children will be multiplicatively fluent in their understandings, the final key indicator is designed to promote these early relationships so that children move beyond the part-whole understanding and consider fractions more broadly. The final key indicator is detailed in Table 3.3.

Table 3.3

Conjectured Key Indicator Five for the Local Instruction Theory

Key Indicator	**Characteristics of tasks**		**Supporting Literature**
	Proposed Fraction Foci	**Proposed Spatial Reasoning Foci**	
Connecting multiplicative relations	**Fraction as an Operator:** Partitive division/ recursive multiplication Times-as-many **Fraction as a Measure:** Part-whole fractions Equivalent fractions **Fraction as a Relation:** Distribution Proto-ratio	Scaling and proportional reasoning to determine equivalent units and proportions. Visual awareness of the relationship between part-part and part-whole quantities	Bruce et al. (2015); Confrey et al. (2014b); Möhring et al. (2015); Noelting (1980); Siemon et al. (2017)

Twenty-two tasks were constructed to develop each of the key indicators and trialled with a participating Year 2 class to confirm the suitability of the tasks for each key indicator and to confirm the appropriateness of the local instruction theory. The pilot was conducted with a small group of children at a time, to examine the suitability of the tasks for inclusion in the intervention program and to examine a range of children's responses to determine what types of mathematical thinking they elicited. The results from this trial enabled the refinement of the key indicators for the local instruction theory and confirmation of the sequence of the tasks in the intervention for the teaching experiment phase. The results from the trial are detailed in Chapter Four. The participants, data sources and analysis techniques are described later in this chapter (Sections 3.5–3.7).

3.4.2 Phase Two: Teaching Experiment

The purpose of this phase was to implement the confirmed intervention program sequence from Phase One with two early years classes and examine the overall coherence and authenticity of the local instruction theory. The structure of the two teaching experiments were as follows. First, I explored the context of each class by observing the children in a series of mathematics lessons taught by the classroom teacher before the intervention began. This process enabled me to identify and analyse children's individualistic mathematical behaviours while considering the social context of children's typical ways of thinking and working mathematically (Cobb & Yakel, 1996) to ensure my pedagogical approach would be appropriate and accessible.

The second element of the teaching experiment phase was to collect pre- and post-assessment data through conducting one-on-one task-based interviews (TBIs) which helped determine what, if any, shifts in children's thinking could be attributed to the intervention program.

Third, the teaching experiment involved fully implementing the intervention program with each participating class. The findings from Class B were analysed, and this informed the redesign and modifications of the intervention program and local instruction theory for Class C. The analysis of each class within this phase are presented in Chapters Five (Class B) and Six (Class C). The participants, data sources and analysis techniques used in this phase are described later in this chapter (sections 3.5–3.7).

3.4.3 Phase Three: Retrospective Analysis

The third phase of DBR is a retrospective analysis where the outcomes of the study are considered to determine both practical and theoretical contributions. The intention of the retrospective analysis phase for authentic DBR should move ‘beyond assessing a method, learning situation or tool and seek to recast the problem at the heart of the intervention…[to] provide the types of epistemic shifting needed for real and sustained change in mathematics education’ (Fowler & Leonard, 2022, p. 17). In the present study, the confirmed local instruction theory provided a basis for highlighting the new theoretical contributions determined from this study about young children’s development of whole number and fraction ideas. The retrospective analysis phase is the basis for Chapters Seven and Eight.

3.5 Participants and School Contexts

This section discusses the study participants and role of the researchers.

3.5.1 Children and Classroom Teachers

This study involved three classes from three separate public schools in regional South Australia. A convenient sampling method was used, whereby a group of participants who volunteer for a study are selected based on their accessibility to the researcher (Fraenkel et al., 2012). Convenience sampling assumes the participants are non-randomly selected based on their ability to meet applicable criteria, such as target age, accessibility or geographical proximity to the researcher, and willingness to participate (Etikan et al., 2016). The limitation of bias can often present in this sampling method due to the conditions in which participants are recruited (Leiner, 2014). To minimise bias, all South Australian Department for Education primary schools within a 50 km radius of my residence were contacted and invited to participate. Five classroom teachers responded to the invitation; however, only three met the criteria regarding the target age group for this study. The participants of this study consisted of 70 children in Years 1 and 2 of primary school and three classroom teachers. Table 3.4 summarises the characteristics of each school and participating class and the role each participant played in the study.

Table 3.4

Summary of Participants' Roles and School Contexts

Class	Preparation phase: Class A (2019)	Teaching experiment: Class B (2019)	Teaching experiment: Class C (2020)
School demographic	Foundation to Year 7 585 children 32 Full-Time Equivalent (FTE) teachers ICSEA score: 998	Foundation to Year 7 588 children 33 FTE teachers ICSEA score: 974	Foundation to Year 7 345 children 21 FTE teachers ICSEA score: 948
Year level	Year 2	Year 1–2 Composite	Year 2
No. of participants	26 children 13 boys and 13 girls Mean age: 7 years, 3 months	23 children 16 boys and 7 girls Mean age: 6 years, 11 months	21 children 11 boys and 10 girls Mean age: 7 years, 2 months
Purpose of involvement	Phase One, Preparation: Trialling tasks and materials	Phase Two, Teaching Experiment: Main study	Phase Two, Teaching Experiment: Main study
Classroom teacher role	Consultation for intervention task design and sequence	Acted as an additional researcher/advisors of task sequence	Acted as an additional researcher/advisors of task sequence

Note. ICSEA = Index of Community Socio-Educational Advantage.

In Australia, each school has an Index of Community Socio-Educational Advantage (ICSEA) score, which indicates a school's educational advantage based on the children's socio-economic background. The median ICSEA score is set at 1,000 with a standard deviation of 100, meaning each of the schools in this study are statistically similar to the median school score of socio-economic advantage and, therefore, demographically similar in educational advantage to each other.

3.5.2 Role of the Researchers

A fundamental role of DBR is the collaborative nature of the research. This methodology involves not just the consultation with a range of stakeholders to help inform the design of the study, but it values the contributions the participating teachers make to the data collection and early analysis. The following section with describe my role and the participating teacher's roles as researchers in this study.

3.5.2.1 Teacher–Researcher

My role for this study was twofold, as I acted as the classroom teacher throughout each iteration of the teaching experiment and the primary researcher. These roles were an intentional component of the research design to ensure consistency in how the intervention program was taught and how the assessment was implemented for data collection. Taking on the roles of both primary researcher and classroom teacher helped to ensure that the forms of knowledge (i.e., content knowledge of fractions and spatial reasoning, pedagogical knowledge for teaching mathematics; Hill et al., 2005) were intentionally the same for each class, while acknowledging the flexibility and adaptability DBR affords in the implementation of an intervention. Previous educational studies have implemented this approach as a measure of trustworthiness for the research (see Carraher et al., 2006; Miller, 2014; Spencer, 2017).

3.5.2.2 Classroom Teachers

The participating classroom teachers were a vital part of developing and implementing the intervention throughout Phases One and Two. In Phase One, the five teachers who responded to the invitation to participate in this study shared their mathematical teaching experiences. I was able to gain insights into how the teachers design and teach their mathematics programs, specifically for whole number and fractions, and the teachers' understanding of spatial reasoning.

I was able to use these insights to compare against current literature to determine how the intervention materials could be constructed and implemented.

The participating classroom teachers for Classes B and C acted as co-researchers within the intervention. As Martens et al. (2019) suggest, 'DBR is a collaboration of researchers and educational practitioners whereby they develop answers to educational problems and advance theoretical understanding' (p. 1204). The classroom teachers in the present study provided their interpretations, reflections, and observations of children's mathematical behaviour throughout the intervention, which was a source of triangulation for the validity of the findings. The perspectives were compared to my own notes, observations, and theoretical perspectives in addition to the children's work samples. The teachers as co-researchers brought their expertise and experiences to the design (Fischer, 2003), which helped to minimise bias as this relationship shifts researchers' focus toward practical design questions, and teachers focus towards a more theoretical perspective on the problem (Kelly, 2006).

3.6 Data Sources

DBR provides the flexibility for various data sources and instruments to be employed. Rather than providing a rigid set of methods, DBR tools and strategies were selected in response to the study's purpose and contexts (Design-Based Research Collective, 2003). In this study, the teaching materials for the intervention and the assessment items for a pre- and post-intervention assessment were the data tools designed for this study. In addition, classroom observations, reflective journals and children's work samples were collected as data sources. Each source is now described and justified.

3.6.1 Phase One: Preparation

The local instruction theory was informed by Confrey et al.'s (2014b) rational number reasoning learning trajectories. Used as a domain-specific theory informing the design of this intervention program, the fraction as relation, fraction as an operator, and fraction as a measure meanings are explored and developed simultaneously. Figure 3.2 represents the underpinning ideas chosen for the purposes of this study for each meaning of fractions (indicated by the inclusion of a red dot) that guided the design of the teaching materials for the intervention program.

Figure 3.2

Underpinning Ideas (Red dots) For the Intervention (Adapted from Confrey et al., 2014b)

[Image removed due to copyright restrictions].

Note. Map of rational number reasoning concepts, adapted from 'Equipartitioning, a foundation for Rational Number Reasoning' (p. 69), In *Learning over time: Learning trajectories in mathematics education*, by A.P. Maloney, J. Confrey and K. H. Nguyen (Eds.), 2014, Information Age Publishing. Copyright 2014 by Information Age Publishing.

The selected ideas in Figure 3.2 span approximately the K–3-year levels. It is reasonable to expect children to work above and below any mathematical topic at any given year level (Goss et al., 2015). Therefore, selecting a wide range of the fraction ideas proposed within this framework helps to accommodate for this variation in capabilities. Table 3.5 defines each underpinning idea within each fraction meaning explored in this intervention.

Table 3.5

Definitions of Fraction Meanings and Underpinning Ideas

Fraction meaning	**Underpinning idea**	**Description and example contextualised for early childhood**
Fraction as Relation	Many-to-one	Many-to-one correspondence is the understanding of part-part relations (*n*:1). (Confrey & Smith, 1995). 'Many' = counterpart objects, 'One' = target object. For example: *Three flowers for each vase* (Sophian & Madrid, 2003). To make a juice mixture, there is a relationship between the water and juice quantities, which may not be equal, but preserved when replicated.
	Distribution	Coordinating units to represent consistent part-part relations, with multiple target objects. For example: *Three children each receive two apples.* In continuous contexts, recognising which part represents more than/less than half. For example, a container with juice and water may have more/less water than juice; the water may be more/less than half of the container capacity. Distribution of parts are proportionally the same (connecting to equivalent fraction idea).
	Proto-ratio	Coordinating two numerical sets additively, typically through *building up* and *building down* strategies (Hino & Kato 2019). Building up: *If there are 6 lollies in one packet, how many lollies in 3 packets?* Building down: *If there are 18 lollies in three packets, how many lollies are sold in one packet?* Continuous example: 4 equal parts water, 2 equal parts juice Comparing the relationship such as 3-quarters water, 1-quarter juice produces a weaker mix than 3-quarters juice to 1-quarter water. Similarly, responses such as: *'3 out of every 12 is the same as 1 out of every 4'* are examples of the proto-ratio idea (Confrey & Maloney, 2015, p. 925).
	Fair Shares	The creation of equal size shares (of discrete collections or continuous wholes) where the shares created exhaust the whole.

Fraction as Operator		Naming fair shares of collections, including counting and relational naming (naming one share in relation to the whole collection or single whole). For example: *12 objects shared among 3 children, a share is 4 objects (per child). Relationally, each child receives $\frac{1}{3}$ of the collection* (Confrey & Maloney, 2015, p. 924)
	Doubling and Halving	Derived from splitting (2-split) which is founded on repeated halving (splits of splits). Perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as *twice as big* for doubling, or *two times as small/half as big* for halving (Confrey & Maloney, 2015).
	Partitive Division/ Recursive Multiplication	Partitive division: Connects to the process of fair sharing by starting with the dividend, then distributing one unit in each set and repeating this process until no more distributions can be shared fairly. For example: *20 stickers shared between 4 children* (Confrey & Scarano, 1995). For young children, *partitive division* often involves trial and error in creating fair shares, rather than problems that involve multiplicative reasoning (Hackenberg & Tillema, 2009). Recursive multiplication: The reversal of equipartitioning is reassembly or recursive multiplication (not counting; Confrey, 2012). For example: The introduction to division and multiplication are viewed as inverse operations to establish a recursive rather than iterative foundation for multiplication (i.e., *times as many*; Confrey et al., 2014b). Involves splitting and reassembly of continuous models and discrete sets.
	Equi-partitioning a single whole	Geometrical reasoning in which symmetries and congruence are utilised to develop equal parts of (primarily) rectangles and circles.
	1-nth-of...	The relationship between '1-nth-of...' in naming *fair shares* and identifying the referent units (greater or less than 1) for the fair shares resulting from equipartitioning (Confrey & Maloney, 2010, p. 973). For example: 12 objects shared among three children, a fair share is four objects (per child). *1-third of 12 is four; 1-and-1-half shares is 6* (Confrey & Maloney, 2015).
	Times-as-many	Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a

		single fair share using ‘times as many’, or ‘times as much’ (Confrey & Maloney, 2015, p. 924). For example: *‘How many times as large as one share is the whole collection? (reassembly)…the whole collection is 3 times as much as one share’* (Confrey & Maloney, 2015, p. 922).
	Geometric Symmetries	Geometric symmetries relate to fair share and equal parts/ groups. ‘When folding, congruence is built directly into the activity through symmetries, but the result of the action is hidden until the paper is unfolded, providing opportunities to examine one’s predictions’ (Confrey, 2012, p. 167).
	Similarity	Related to splitting through identifying similarities between the properties of equal shares and non-symbolic proportional relationships (continuous parts and sets).
	Scaling	Related to *times as many*. ‘There is only one salient dimension here, namely, objects. The splitting operation in this instance establishes the foundation of the ideas of a scalar (a dimensionless number possessing only magnitude) and a scaling factor’ (Confrey, 2012, p. 162).
Fraction as Measure	Measure	Directly related to fair share, in that when fair shares are created, these shares represent a quantity that can be used as a *measure* in comparison to the whole.
	Many-as-one	Many-as-one is a group of *m* objects, where the quotient represents the extensive quantity that one sharer receives (Confrey, 2012) For example: *If I share 12 lollies with my friend, we each get 6 lollies, six is a fair share.*
	Composite Units	Derived from splitting, a composite unit is a unit of units. E.g., $\frac{3}{4}$ is a composite unit of three, $\frac{1}{4}$ units as a result of a from a three split, 1 $\frac{1}{2}$ is a composite of two $\frac{3}{4}$ units derived from halving; related to recursive multiplication.
	Unit Fractions	Unit fraction involves identifying and naming a single share of *n* fair shares as ‘1-nth-of... 1’ (Confrey & Maloney, 2015).
	Part-Whole Fractions	The conceptualisation of the relationship between *measure, many-as-one, composite units,* and *unit fractions* for 1. For example: *An apple is cut into y equal parts and x of these parts are eaten* (Tsay & Hauk, 2009).
	Equivalent Fractions	The equivalence of two fractional parts

	For example: $\frac{2}{4} = \frac{1}{2}$ (Confrey & Maloney, 2015). Explored non-symbolically via geometric models and discrete sets in this study.

In addition, the local instruction theory is developed on the conjecture that a spatial reasoning approach is critical for children to make sense of the various ideas and meanings of fractions identified above. Specifically, spatial visualisation and spatial proportional reasoning were identified in Chapter Two as spatial constructs that are highly desirable for promoting this novel approach. Table 3.6 describes each of these spatial constructs based on the literature review, contextualising these in relation to the fraction ideas identified above.

Table 3.6

Spatial Reasoning Constructs and Their Relationship to Rational Number Reasoning

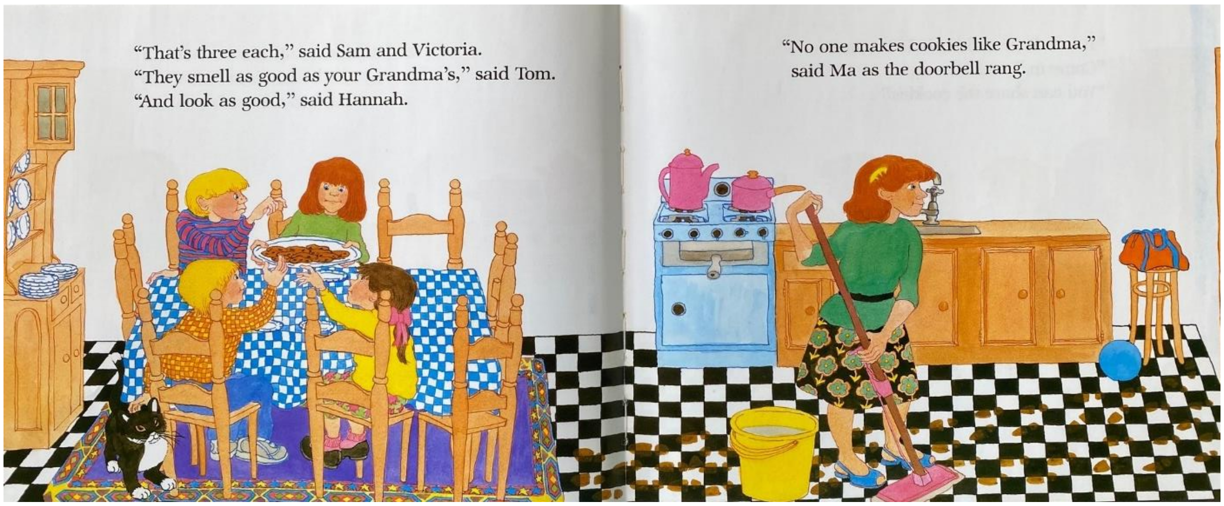
Spatial reasoning construct	Definition of spatial construct and relationship to rational number reasoning
Spatial Visualisation	Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations *within* objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999). The intent of this spatial construct for this study is to develop children's visualisation capabilities in relation to partitioning, unitising and equivalence concepts in a range of discrete and continuous contexts. For example, children will be encouraged to visualise the size and shape (*geometric symmetries* and *similarities*) and arrangements (*composite units, part-whole*, *many-to-one* or *many-as-one*) of a *fair share* by mentally manipulating and transforming objects or sets of objects. In addition, the process of creating *a fair share* (e.g., visualising *partitive division/recursive multiplication, doubling/halvin*g) of one quantity can be visualised and compared to another related quantity (*fraction equivalence, part-whole*) or unrelated quantities (*measure*) as examples. Spatial visualisation can also enable children to visualise and mentally manipulate the increase/decrease in fair shares or units for *distribution* (or redistribution) and *proto-ratio* ideas.

	As described in Chapter Two, *mental rotation* is referred to as a subset of spatial visualisation and is the ability to imagine how an object would look if it were rotated on an axis; that is, to mentally turn a 2D or 3D object (Frick et al., 2013). *Mental rotation* can enable children to visualise how different parts are related to each other, by rotating parts to explore their size and geometric attributes when considering fraction magnitude.
Spatial Proportional Reasoning	Spatial proportional reasoning is the non-symbolic, visual recognition that shape, object, and arrangements of different wholes can have the same value and therefore are *equivalent.* This can be an awareness of *doubling and halving*, *times as many*, *distribution*, and *proto-ratio* ideas in the development of fraction magnitude. Spatial proportional reasoning includes *scaling*, which refers to the ability to compare different-sized spaces (Frick & Möhring, 2016); the ability to relate distances in one space to distances in another space (Frick & Newcombe, 2012). *Spatial scaling* and *proportional scaling* recruit overlapping cognitive processes (Möhring et al., 2018); therefore, *spatial proportional reasoning* for the purposes of this thesis includes the ideas of both spatial scaling and non-symbolic proportional reasoning. At this age, it is the perceptual awareness of this relationship rather than necessarily quantitative measures. For example, transforming one space in size to match the other (Frick & Newcombe, 2012), such as 'mentally shrink[ing] or expand[ing] spatial information in the sense of zooming in or out (of the map) …internally transforming magnitude information' (Möhring et al., 2018, p. 58). Thus, *geometric symmetries* and *similarity* of spaces, objects and arrangements are key connections in the development of fraction understanding.

The tasks were based on the context of two picture storybooks: *The Doorbell Rang* by Pat Hutchins (1989) and *Knock, Knock Dinosaur!* by Caryl Hart (2017). Research indicates that picture storybooks can be a powerful tool for learning mathematics, particularly with younger children (Marston, 2010). Picture storybooks can engage children in unfamiliar and familiar mathematical ideas, enabling mathematical thinking and curiosity to explore more formal levels of understanding (Van Den Heuvel-Panhuizen et al., 2009). *The Doorbell Rang* (Hutchins, 1989) was chosen as it provides a context to engage children in the ideas of fair sharing/partitioning and distribution (see Figure 3.3).

Figure 3.3

The Doorbell Rang Excerpt (Hutchins, 1989)



Note: Example of text from *The Doorbell Rang*. Reprinted with permission.

During the story, the children need to share 12 cookies fairly, but as more people arrive to the house, the children must work out how to redistribute the cookies among the changing number of people.

Knock, Knock Dinosaur! (Hart, 2017) is a story about a boy receiving a delivery of toy dinosaurs that are actually *life size*. This introduces a context to explore the ideas of ratio and proportion (see Figure 3.4).

Figure 3.4

Knock, Knock Dinosaur! Excerpt

Note. Example of text from *Knock! Knock! Dinosaur*. Reprinted with permission.

The tasks for the intervention were designed based on one of the picture story books. An example of the tasks is illustrated in Table 3.7. The full suite of tasks designed for the perpetration phase pilot is in Appendix A.

Table 3.7

Pilot Task Example

Pilot Task 1: Sharing Cookies				
Fraction Foci		**Spatial Reasoning Foci**	**Children trialling the task Class A**	
Fraction as Operator	**Fraction as Measure**	**Construct**	**Group 1**	**Group 2***
Fair Share *Partitive division/ recursive multiplication*	*Many-as-one*	*Spatial Visualisation*	Child 1 Child 2 Child 3 Child 4 Child 5	Child 6 Child 7 Child 8 Child 9 *Child 23 & Child 25 (introduced later in the pilot for gesture observations—see Chapter Four)

Relationship between fraction ideas and spatial constructs:
Visualising partitive division/recursive multiplication between parts/ shares and whole. Conceiving the change in size of share as more shares are required. Visualising shares involving mixed numbers.

Task:
Introduce the picture book—*The Doorbell Rang* by Pat Hutchins. Ask the children to describe what is happening in the story.
Each child receives a 'story board' that shows how many children were at the table at each part of the story. The children are asked to model/ draw how each group of cookies would be shared in each of the boxes.
Story board (A3 size):

12 cookies, 2 children	12 cookies 4 children
12 cookies 6 children	8 cookies 12 children

Children are provided with paper circles (as cookies) and plastic counters if they choose to use them.
Children are asked to name how they might describe the different shares of cookies.

3.6.1.1 Work Samples

Representations are a critical element of learning mathematics (Bobis & Way, 2018); therefore, children's work samples were collected during this study. The work samples provide insights into children's mathematical thinking (Cartwright, 2019), which were analysed in addition to the researchers' observation notes to make assumptions about how children understand the various fraction ideas. From a Vygotskian perspective, interpreting children's work samples is a critical element for understanding young children's concept development (Woleck, 2001), particularly the act of 'drawing-telling' (Wright, 2007) where children create and share meaning in verbal and non-verbal modes (MacDonald & Lowrie, 2011). An A3 blank workbook was provided for each child in this study, where they could record their problem-solving strategies and represent their understandings for each task.

3.6.1.2 Field Notes

During each small group session with the children in Class A, I took extensive field notes on my observations on how children engaged with the tasks. This included capturing the children's descriptions they provided during the tasks, observations on how children manipulated various concrete models or how they constructed pictorial and diagrammatic representations, and any associated gestures they used throughout the tasks. Field notes enabled me to determine the suitability of the tasks as well as identify specific forms of mathematical thinking that the tasks promoted.

3.6.2 Phase Two: Teaching Experiment

The teaching experiment conducted with Classes A and B used the data sources of pre-intervention classroom observations, pre- and post-TBIs, lesson plans from the intervention program, field notes and children's work samples. Each of these is described and justified below.

3.6.2.1 Pre-Intervention Classroom Observations

An interpretivist perspective requires an understanding of the classroom context in which the intervention is explored to determine if and how the parallel approach was successful. The participating teachers for Classes B and C of the teaching experiment were observed teaching mathematics in their classrooms prior to the intervention commencing. The observations took place in the scheduled mathematics lessons of each class. I observed three lessons per class on the mathematics topic the teacher had programmed. It was a participation requirement that children had received no instruction on fractions within the same school year prior to the intervention. This condition for participating was to mitigate confusion between the intervention program and what the children may have experienced as part of their typical fraction program, given that the intervention was novel in design. The pre-intervention classroom observation enabled me to understand the social context of the classroom.

3.6.2.2 Pre- and Post-Assessment: Task-Based Interviews

A one-on-one, TBI was developed as a pre- and post-intervention assessment in this study to determine the changes in children's understanding that could be attributed to the intervention. TBIs are typically used in mathematics education to enable judgements about a child's existing mathematical understanding, growth or changes in knowledge, and their use of representations and ways of reasoning (Goldin, 1997; Maher & Sigley, 2020). TBIs can be structured, semi-structured or open, depending on the purpose of the research. As this study was interventionalist in nature, the TBI was semi-structured. I provided the verbal instructions for each task, prompting and questioning the children to uncover the reasons for their responses and representations. The children did not receive feedback on their results.

The TBI for this study consisted of 24 items. The assessment was organised in three sets: Set One assessed aspects of children's whole number knowledge, such as subitising and part-

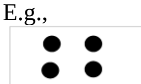
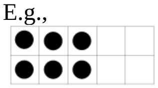
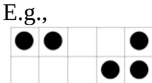
part-whole number knowledge; Set Two assessed children’s place value understandings; and Set Three assessed children’s fraction understanding and spatial reasoning capabilities.

The first nine items, divided into two sets, Set One: Trusting the Count and Set Two: Place Value, were derived from the Assessment for Common Misunderstandings (AfCM) resources from the Department of Education Victoria (Siemon, 2006). The whole number items were included for two reasons: (1) to determine children’s typical whole number knowledge to use as a benchmark of their mathematical understandings, and (2) to determine if the inclusion of a spatial reasoning approach to learning fractions had any impact on the children’s whole number knowledge.

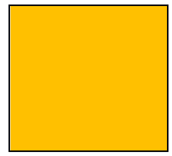
Set Three: Fractions and Spatial Reasoning comprised of items 10 to 24. These items were designed to assess children’s knowledge of and ability to reason with fraction as a measure, fraction as an operator, and fraction as a relation meanings (Confrey et al., 2014b). Similarly, the spatial constructs of spatial proportional reasoning and spatial visualisation were also assessed in a range of items throughout the assessment. These were the primary spatial constructs underpinning the intervention, determined from the literature review. Examples of the TBI items are presented in Table 3.8. The full TBI is located in Appendix B.

Table 3.8

Pre- and Post-Assessment Items for the Task-Based Interview

Set One: Trusting the Count			
Item	**Focus**	**Materials/Stimulus for each Task**	**Source and Description**
1: Subitising collections	*Part-part-whole knowledge*	Cards 1–6 (common dot die arrangement) E.g.,  Cards 7–10 (ordered arrangements) E.g.,  Cards 7–19 (ten frames ordered and random) E.g., 	Hold the Subitising Cards out of the view of the child. Show each card in order for approximately 2 seconds. After each card, ask, *'How many dots were on that card? … How did you work that out'* (Siemon, 2006)
2: Hidden counters task	*Part-part-whole knowledge*	Nine counters Calico bag (Example of resources provided)  	Place five counters and bag in front of child, rattle to demonstrate that there are counters in the bag. Place four counters in front of child. *'There are four counters here and five more in this bag. How many counters altogether? How did you work that out?'* (Siemon, 2006)
Set Two: Place Value			
Item	**Focus**	**Materials/ Stimulus for each Task**	**Source and Description**
5: 26 counters	*Place value parts, Composite, countable units*	26 counters plastic counters (Example image of resource provided) 	Tip out counters and say, *'Please count these and write down how many'*. Circle the '6' in '26' and ask, *'Does this have anything to do with how many counters you have there?'* Circle the '2' in '26' and repeat the question. Ask the child to explain their thinking if not obvious. (Siemon, 2006)
6: Place-value Bundles	*Place value parts Composite, countable units*	13 bundles of 10 pop sticks and 16 single sticks (Example image of physical resources provided) 	Place bundles and single sticks in front of the student, point to bundles and say, *'Do you know how many straws are in this bundle?*' Suggest counting, if necessary, then say, *'Please show me how you would make 34'*. If child asks or moves to unbundle a 10, point to the bundles of 10 and say, *'Before you do that, is there any way you could use these to make 34?'* (Siemon, 2006)

Set Three: Fractions and Spatial Reasoning

Item	**Focus**	**Materials/ Stimulus for each Task**	**Source and Description**
10: Folding Fractions	*Fair share* *Doubling/ halving* *Unit fractions* *Spatial visualisation*	Image of a square: 	Child is shown the image of a square. *'How many ways can you imagine folding a square in half? Can you describe what you think it would look like if you folded it in half, then in half again? What is each part called?'* (University of Cambridge, 1997-2023)
11: What fraction is green?	*Composite units* *Unit fractions* *Distribution (two parts white to three parts green)* *Spatial proportional reasoning*	Image of rectangle: 	Child is shown the rectangle and asked, *'What fraction of this rectangle is shaded green? How did you work that out?'* (Created by researcher)
23: Plant growth rate	*Distribution* *Proto-ratio* *Doubling and halving* *Spatial Proportional Reasoning*	Image provided of the plants as a distractor:  Plant A  Plant B	Comparing rate of growth. *'If plant A grows 5 cm in half a year, and plant B grows 8 cm in a whole year, which is growing faster? How do you know?'* (Adapted from Dole et al., 2012)

The children were provided with a booklet where they could record their answers to each question. The TBI took approximately 25 minutes per child, and I recorded (via pen and paper) all interactions (in line with ethics provisions). For example, if a child used materials to solve a question, I recorded which materials, how they used these materials, any associated gestures they used, and documented their explanations.

3.6.2.3 Lesson Plans

As described in the preparation phase, the tasks were analysed and organised into a coherent sequence of lessons for the intervention. Each daily lesson was structured using the *Launch–Explore–Summarise* cycle (Sullivan et al., 2015). This three-phase model provides a structure for teaching and learning mathematics in the early years of schooling, particularly for introducing complex tasks (Sullivan et al., 2015). The *Launch* phase involves sparking children's curiosity about the topic by providing some targeted questions or a provocation—such as the picture books described above. This phase introduces the underpinning ideas and problems without the teacher 'telling' the children how they will solve them. The *Explore* phase allows children to work individually and collaboratively on complex problems, developing strategies and sharing ideas. Finally, the *Summarise* phase is where intentional teaching occurs as children reflect on their thinking and working and use this as a basis for further exploration and problem posing. Intentional teaching is an underpinning principle of early childhood pedagogy and, therefore, a critical component of each lesson's summarise phase. It is a deliberate and purposeful opportunity to extend children's thinking about fractions and deepen their understanding of these ideas and strategies (Department of Education, Employment and Workplace Relations [DEEWR], 2009).

Each lesson (which included several related activities throughout the *Launch*, *Explore*, and *Summarise* phases) ran for approximately 60 minutes and was delivered over 13 consecutive school days. The complete intervention program describing each lesson is in Appendix C. Each lesson plan identified the fraction ideas of focus within the nominated fraction meanings. For example, in Lesson 1, the activities were designed to target several fraction as an operator and fraction as a measure ideas. Similarly, the spatial reasoning construct is described in terms of how it is intended to support the learners in each lesson (see Table 3.9).

Table 3.9

Intervention Program Lesson Plan Examples

Lesson 1: Sharing Cookies		
Fraction Foci	**Spatial Reasoning Foci**	**Relationship between fraction ideas and spatial constructs**
Fraction as Operator *Fair shares* *Doubling/ Halving* *Partitive division* *1-nth-of……* **Fraction as Measure** *Many-as-one*	**Spatial Visualisation**	Visualising partitive division/recursive multiplication between parts/shares and whole. Conceiving the change in size of share as more shares are required. Visualising shares involving mixed numbers.

Launch

Questions and provocations for the children: *When you hear the word half, what do you think of? (close your eyes and imagine). If I asked you to imagine what this strip of paper (20 cm x 10 cm) would look like if it was folded in half, what do you see? What do you imagine? How do you know you have folded it in half?*
When you hear quarter, what do you see in your mind?
Draw the pictures you see in your mind about half and quarter or anything you know about fractions (the children are given individual whiteboards to work on)

After children explore these questions and representations, they will share with each other in small groups.

Explore

Picture book: *The Doorbell Rang* by Pat Hutchins. Ask the children to discuss and describe what is happening in the story.

Each child receives a ‘story board’ that shows how many children were at the table at each part of the story. The children are asked to model/draw how each group of cookies would be shared in each of the boxes.

Example of A3 story board

12 cookies, 2 children	12 cookies 4 children
12 cookies 6 children	8 cookies 12 children

Children are provided with paper circles (as cookies) and plastic counters if they choose to use them.
Children are asked to name how they might describe the different shares of cookies.

Summarise

Discuss how the children problem solved; specifically, *8 cookies between 12 children* (a complex problem not explored in the picture book)

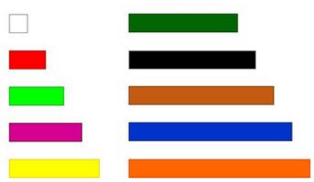
*Intentional Teaching: *How much of the whole set of cookies (12) does one of these eight children have? What patterns do you notice about the shares created?*
What happened to the number of cookies each person receives when there are more children to share the cookies with?
What did you notice about each person's share? What does this have to do with fractions? (naming/ partitive division, 1-nth-of...… for fractional parts)

Lesson 10: Dinosaurs (Part 3)

Fraction Foci	Spatial Reasoning Foci	Relationship between fraction ideas and spatial constructs
Fraction as Operator *Doubling and halving* *1-nth-of... …* *Times as many* *Similarity* *Scaling* **Fraction as Measure** *Composite units* *Unit fractions* *Part-whole fractions*	**Spatial Visualisation** **Spatial Proportional Reasoning**	Estimating fractional lengths of paths on carpet maps. Paths are not straight, so children need to engage in spatial visualisation and mental rotation to compare the length of multiple paths and use spatial proportional reasoning to estimate measures within a single pathway/region. Creating scaled representations of fractional paths, naming and describing the distributions of the measures: half of …path is greater than half of (another) path.

Launch

Task: *Cuisenaire Fractions.* To engage children in spatial proportional reasoning for comparing fractional parts.



Children are provided with sets of Cuisenaire rods and cards with a range of questions, such as:
If this is one (orange, dark green, etc.)—*which rod is half?*
If this is 2-thirds (dark green), what is one whole?
What is three times light green?
What is four red the same as? What is the relationship?

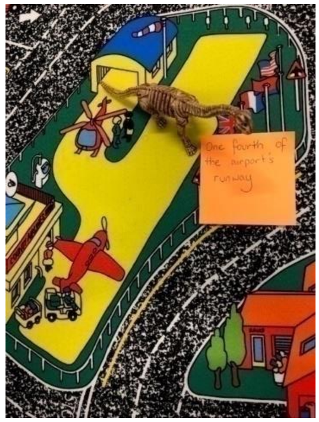
Explore

Revisit picture book *Knock, Knock Dinosaur*! by Caryl Hart.

*Preceding this lesson: *The dinosaurs have escaped the boy's house! They've decided to explore the neighbourhood—here is the map.*
The postman said they saw a T-Rex halfway between the boy's house, and the zoo. Where could it be? A delivery driver said she saw a dinosaur halfway between the central fountain and the duck pond—where would that be? (museum). *A pilot saw another dinosaur 2-thirds of the way along the road in front of the café, heading toward the food market…where would this dinosaur be?*

Children were provided with large carpet maps of different 'towns and a set of clues describing where several dinosaurs were seen, unique to each map. The children used small plastic dinosaurs and sticky notes to place on the mats and solve each clue.
Task: Children represent parts of their carpet map explored in the previous two lessons, which show where each dinosaur was located. They need to draw the points of interest (e.g., the runway of the airport), draw the position of the dinosaur and then write

in words their explanation (e.g., ‘dinosaur spotted *1-quarter* of the way along the runway’). Examples of carpet maps are included below.



Children can draw a map (or part thereof) of another’s groups town, again describing the position of the dinosaur using *times as many, double/half* and *unit* or *composite unit fraction* terms or phrases and gestures that suggest these.

<u>Summarise</u>
*Intentional Teaching: *What is the same about the carpet map (zoo/airport/farm, etc.) and your map?* (same proportions/fraction, different scale, etc).
What is different? (absolute size).

Gallery walk: *Check out the other group’s positions of their dinosaurs. Do you agree on their position based on their task cards? Is there a different position the dinosaur could have been standing?* (i.e., one-third of the runway depends on which end of the runway is considered the ‘start’).
What was hard about this task? What strategies did your team use to work out the position of your dinosaurs?

Crucial to the design of this study, the intervention program replaced the classroom teacher’s mathematics program the children would have otherwise experienced. That is, the participating children in Classes B and C did not receive any additional mathematics lessons during the intervention period. This condition was to enable the findings of the study to be directly attributed to the intervention.

3.6.2.4 Field Notes: Teacher and Researcher Reflective Journals

Each of the participating classroom teachers of Classes B and C acted as additional researchers for the study. As part of this role, the classroom teachers and I kept separate reflective journals to document our observations throughout each lesson. The purpose of documenting our observations was to provide thick descriptions (Creswell, 2003) of the context, to capture children’s use of language, how they engaged in various tasks and used the provided materials and compare our insights and interpret how they were developing their understanding throughout the intervention (Creswell, 2013). The field notes also provided a form of triangulation, discussed later in this chapter.

Before the intervention, I met with each participating teacher to discuss the nature of the observations. The observations included making notes on children’s use of materials, specifically focusing on documenting how they constructed their representations and what, if any, gestures accompanied their discussions or engagement during whole class, individual and small group contexts. Classroom teachers were encouraged to ask clarifying questions of the children, if needed, during the observation and record the interaction to contextualise the learning situation.

3.6.2.5 Work Samples

As discussed in the data sources relevant to Phase One (Section 3.5.1), the teaching experiments in Phase Two also used work samples as a source of data. An A3 blank workbook was provided to each child for this study, where they could record their problem-solving strategies and represent their understandings in each lesson. In addition, the children were provided with an assessment booklet to record their answers or represent their thinking for the pre- and post-intervention TBI items.

3.7 Data Analysis

The data analysis process occurred throughout all three phases of this study. Thematic analysis was the primary method of analysis used in this study because it is driven by observation and interpretation (Scharp & Sanders, 2019). Quantitative analysis techniques were also employed. This section describes the data analysis methods in relation to the data sources.

3.7.1 Pilot Task Analysis

The tasks designed for the intervention program were piloted as part of the preparation phase with small groups of four to six children at a time (see Appendix A). The intention of trialling these tasks with children was to evaluate how a child may engage with the ideas, the level of difficulty the task presented and the scaffolding that might help children develop their knowledge and understanding. Specifically, Posner et al.'s (1982) framework for describing conceptual change informed this analysis. The conditions for conceptual change are *intelligibility*, *fruitfulness,* and *plausibility*. To summarise, Treagust and Duit (2008) state:

> An *intelligible* conception is sensible if it is non-contradictory, and its meaning is understood by the student; a *plausible* conception is considered believable in addition to the student knowing what the conception means; and the conception is *fruitful* if it helps the learner solve other problems or suggests new research directions. (p. 299)

The theory of conceptual change intention is to 'understand how the components of an individual's conceptual ecology interact and develop and how the conceptual ecology interacts with experience' (Strike & Posner, 1992, pp. 155–156). This assumption implies that the framework is used to analyse children's learning development over a sustained period of time. However, for the purposes of this study, the framework is used to analyse the adequacy of the tasks and the potential they offer to children for sufficient conceptual change as a result of the intervention. Table 3.10 details the conceptual change criteria.

Table 3.10

Criteria of the Conceptual Change Framework

Condition	Description	Example of analysis Task: *Share 12 cookies between eight people fairly*
Intelligibility	Children understand the context in ways that are familiar to them and in what context. For example, intelligibility requires the learner to make sense of the problem and indicate possible thought patterns and strategies that indicate the child understands the context and content of the problem—even if they cannot solve it at this point.	While many children across the pilot could not initially move from partitioning discrete collections to continuous in the same context (that is, 12 shared between eight results in 1-and-a-half cookies each), the children recognised that there needed to be equality achieved in the shares created for each person, even though they were not immediately sure how this could be achieved.
Fruitfulness	Learners abandon their original ideas and are now able to solve problems that were *not* previously solvable for the learner or helping provide new ways of examining a situation.	Children understand sharing/partitioning can occur in both discrete and continuous sets simultaneously. This component of analysis for the tasks and activities also targets the spatial reasoning constructs emphasised in the lessons, because fruitfulness is about helping the learner solve problems they could not previously solve before. That is, using spatial visualisation enabled many children to consider alternative strategies for sharing parts of cookies (not just sets of whole cookies).
Plausibility	The concept and new ideas must coordinate with learners present understanding without any conflict. Children produce predictive understandings that support their strategy choice and how this will in fact result in a true or plausible outcome.	Children's ability to estimate, visualise and predict the outcome of a problem, such as recognising that 12 cookies *can* be shared equally between eight people, and, therefore, fractions can be represented as mixed numbers (not just proper fractions).

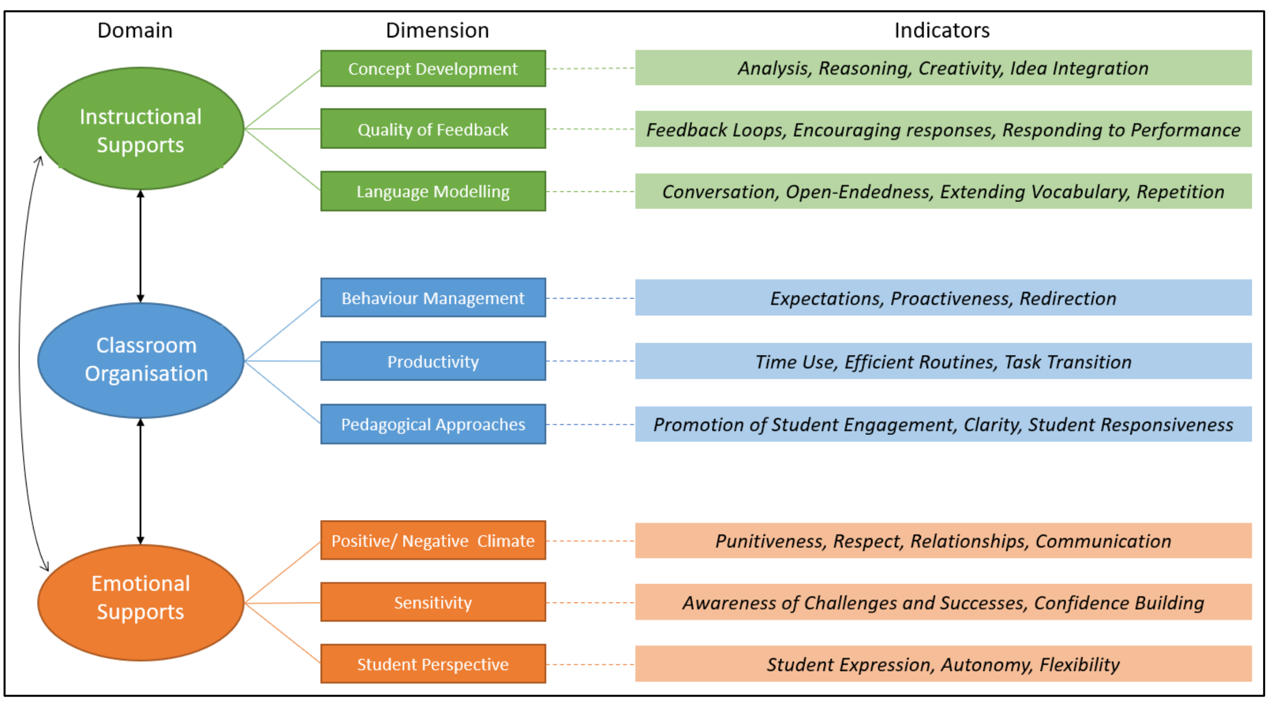
The findings from the pilot and how the results informed the creation of the intervention program and refinement of the local instruction theory for Phase Two (the teaching experiment) are presented in Chapter Four.

3.7.2 Classroom Observation Analysis

Hamre and Pianta's (2007) Components of the Classroom Assessment Scoring System (CLASS) framework was used to analyse my classroom observations prior to the Class B and C interventions. This empirically supported and theoretically driven framework allowed me to analyse the teacher–class interactions in three major domains: *emotional supports*, *classroom organisation* and *instructional supports* (Hamre et al., 2009). These three domains are further separated into dimensions, with indicators that help synthesise and organise the themes that best describe the environment, depicted in Figure 3.5 (Hamre et al., 2009).

Figure 3.5

Classroom Assessment Scoring System Framework (Pinta & Hamre, 2009)



The intention of this framework is to focus on *teacher–children* interactions at a classroom level, not at the level between the teacher and individual children. Typically, each of the domains are rated on a seven-point scale—low (1, 2), moderate (3–5) and high (6, 7)—and assessed by pairs of observers during different time points throughout a lesson to establish inter-rater reliability. This framework was implemented in a modified form, whereby I analysed my written observations against these themes to determine the typical *emotional*, *classroom* and *instructional supports* employed by the teacher and how this affected the children's learning experiences. Praetorius and Charalambous (2018) state that it is important that an observational framework provides a comprehensive picture of the quality of instruction, and thus the elements within a specific framework must be carefully considered with regard to the intended study. For the present study, the CLASS framework provided the structure required to interpret the typical mathematics classroom, which helped inform both pedagogical and organisational aspects of the intervention.

3.7.3 Pre- and Post-Task-Based Interview Analysis

All pre- and post-TBI assessment items were scored using rubrics to assess children's mathematical competencies and types of reasoning at the beginning and immediately after the intervention. Examples of the scoring rubrics for the whole number items (Set One and Two) were derived from the Assessment for Common Misunderstandings tools (Siemon, 2006; see Table 3.11).

Table 3.11

Example of Task-Based Interview Rubric

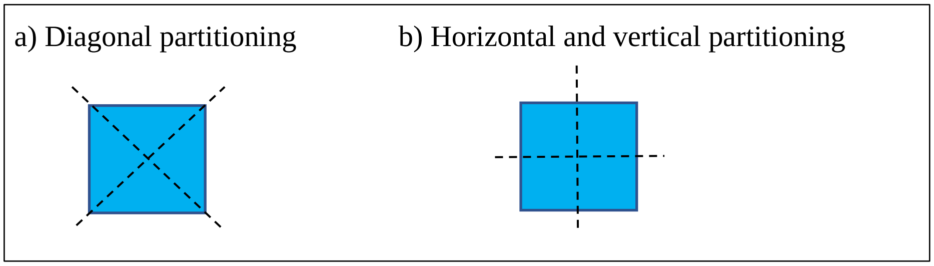
Set One: Trusting the Count Scoring Rubric			
Item	No/incorrect response	Partially correct response and reasoning	Correct response and reasoning
1: Subitising collections	Little/no response (e.g., identifies first card of each set only), or clearly guessing.	Consistently recognise numbers up to five in two seconds or less, can occasionally recognise some numbers larger than five in two seconds May recognise some teen numbers without counting on by ones.	Consistently recognises numbers up to 10 in two seconds or less. Recognise all teen numbers without counting on by ones/ demonstrating part-part knowledge (conceptual subitising).
2: Hidden counters	Little/no response (e.g., identifies first card of each set only), or clearly guessing.	Counts the five that can be seen and makes some attempt to count the hidden collection by counting on or counting all but unable to complete or incorrect.	Immediately correct on the basis that 'I just know' use of number fact knowledge.
Set Two: Place Value Scoring Rubric			
Item	No/incorrect response	Partially correct response and reasoning	Correct response and reasoning
5: Counting 26 counters	Little/no response to most cards.	Counts by ones and records 26 but may not recognise significance of '2' and '6' in '26'. May say 'twenty' in response to the role of 2 in 26 and identify one ten more.	Efficiently identifies 26 as two tens and six ones.
6: Place-value Bundles	Little/no response. Counts bundles and ones at random.	Makes 34 using tens and ones, but not in a way that suggests 10 ones is understood as one ten.	Makes and/or records 34 using bundles (or explaining the representation of three tens and four ones) efficiently and accurately.
Set Three: Fractions and Spatial Reasoning Scoring Rubric			
Item	No/incorrect response	Partially correct response and reasoning	Correct response and reasoning
10: Folding Fractions	Unable to respond. Incorrect solution for forming half (e.g., unequal parts created).	Child could identify at least one way the square could be partitioned in half. Does not discuss the equality of the parts (even with prompting).	The child recognises at least four ways the square could be partitioned in half. Understands the need for parts to be equivalent
11: What fraction is green?	Unable to respond. Responds with a whole number/counts the parts (such as one	Recognises the proportional difference in size between the green rectangle and white rectangles. May use a	Articulates 3-fifths, reasoning that the white parts, although separated by the green part,

	green part, two white parts).	benchmark like half to reason between each part.	have a direct relationship to the whole.
23: Plant growth rate	Unable to respond or incorrect answer.	Indicates awareness of different time period and the effect this has on determining the growth rate, but does not apply it to the context of the plants	Recognises that the size of the plants needs to be considered during the same period of time and uses doubling/ halving knowledge.

An answer was scored partially correct if the child demonstrated a procedure or understanding that could lead them to a correct answer or reasoning. However, they may have made other errors while recording their answers. Alternatively, the item may have multiple possible answers, but a child only recognises some of them. An example of a partially correct score is for Item 10: Folding Fractions. The children are asked to look at an image of a square and visualise how many ways that square could be halved. I anticipated that the children would respond in two ways only: either two folds diagonally (see Figure 3.6a) or a vertical and horizontal fold (see Figure 3.6b). These were expected as these are likely the most obvious or familiar experiences children may have had with folding.

Figure 3.6

Expected Answers for Item 10: Folding Fractions



The purpose of using a rubric and taking written observations during each question was to develop a fine-grained analysis of what skills, understandings, or difficulties each child possessed prior to and immediately after the intervention.

3.7.3.1 Task-Based Interview Quantitative Analysis

A two-sample paired sign test (Cohen, 2013) was conducted on the TBI data to determine if there were any significant changes in children's pre- and post-assessment responses. This test is typically used to determine the significance of the change (if any) that has occurred between pre- and post-measure items. Each child's response to each item in the post-measure is paired with their response to the corresponding pre-measure item (Cohen, 2013). This ensures that the difference between the pre- and post-assessment responses can be attributed to the intervention, and not to differences between the individuals taking the test (Howell, 2010).

This test was chosen because it applies to data that are non-parametric, discrete data, such as that generated by the TBI, as opposed to the sort of data that is required for other pairwise sign tests (such as the Wilcoxon signed-rank test or analysis of variance test [ANOVA], (Cohen, 2013).

For this study, three pairwise outcomes were possible for each item on the pre- and post-TBI:

- 'Positive Change'—either a partially correct response paired with a correct response, or an incorrect response paired with a partially correct or correct response.
- 'Negative Change'—a correct response paired with a partial or incorrect response, or a partial response paired with an incorrect response.
- 'No Change'—the same response, regardless of whether or not the response was correct, partially correct, or incorrect.

This created nine possible pairwise outcomes for each item, as shown in Table 3.12.

Table 3.12

Possible Change Outcomes for the Task-Based Interviews (TBI)

Possible Change Outcomes		Post-TBI		
		Correct	**Partially Correct**	**Incorrect**
Pre-TBI	**Correct**	No Change: Negative	Negative	Negative
	Partially Correct	Positive	No Change: Negative	Negative
	Incorrect	Positive	Positive	No Change: Negative

As the two-sampled paired sign test requires a binomial calculation based on the probability of a positive change outcome versus a negative change outcome (Sarty, 2020), a 'no change' response in this study was regarded as evidence of a negative change, even though this may have been the result of a child obtaining correct responses on both pre- and post-tests. Considering the context of the intervention program, the age of the children and the pilot findings, it was deemed highly unlikely that children would perform in this way. Based on this, only three of the nine possible pairwise outcomes were positive, giving a 33% probability of a positive change.

For the purpose of determining overall change, the pairwise outcomes for each item for Set One and Set Two were combined. This meant that the total number of responses for each set was determined by multiplying the number of children by the number of items in each set. For example, in Class B (23 children), for Set One (four items) there were 23 x 4 = 92 possible responses. Table 3.13 provides an example of this analysis for Class B, Set One.

Table 3.13

Example of Paired Sample Sign Test: Analysis of Class B, Set One

TBI set	Items	Total possible responses	Positive change	Negative change	*p*-value
Set 1: Trusting the count	1, 2, 3, 4	92	40	52	0.016

Note. TBI = task-based interview.

For Set Three, the items were grouped by fraction meaning and spatial reasoning category. Some items were considered in multiple categories. This can be seen in Table 3.14.

Table 3.14

Set Three Item Description

Category	TBI items
Fraction as a measure Ideas	10, 11, 12, 15, 16, 20 21 22, 24
Fraction as an operator Ideas	10, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24
Fraction as a relation Ideas	11, 13, 14, 23
Spatial Visualisation	10, 12, 15, 16, 17, 18, 20, 21, 22, 24
Spatial Proportional Reasoning	8, 11, 13, 19, 23

Note. TBI = task-based interview.

The results of the pairwise sign test for the TBI sets are described for each class in the teaching experiment phase, discussed in Chapters Five and Six.

3.7.4 Thematic Analysis

Thematic analysis is an approach described as ‘a search for themes that emerge as being important to the description of the phenomenon’ (Fereday & Muir-Cochrane, 2006, p. 82). Thematic analysis provides researchers with a method for identifying patterns of meaning within the data that the researcher deems to be important concerning the research questions (Braun & Clarke, 2006, 2013; Daly et al., 1997). In the present study, all data sources were thematically analysed; that is, the daily field notes from the classroom teachers' and researcher's journals, the work samples from each lesson, the pre- and post-intervention TBI data (including children's work samples and my observations of their behaviour and discussion) and the pre-intervention classroom observations were examined to identify common ideas, patterns and themes that were used to interpret how the children developed their understanding throughout the study.

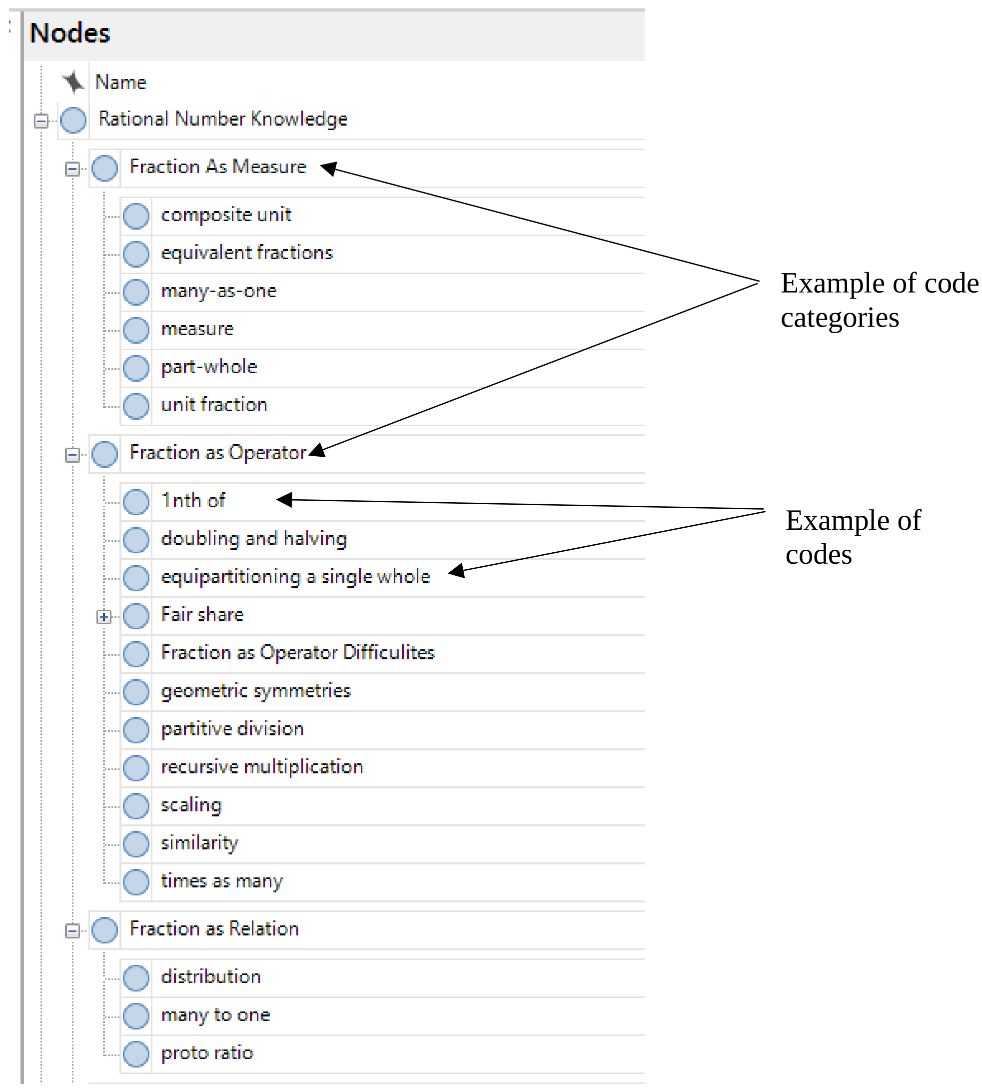
The data were coded to determine patterns in meaning and behaviour. A code is a short, succinct description of a phenomenon a researcher wants to explore or understand. A theme is 'a pattern in the information that at minimum describes and organises the possible observations and at maximum interprets aspects of the phenomenon' (Boyatzis, 1998, p. 161). Themes can be represented as subthemes or main themes, depending on their prevalence throughout a corpus of data. In addition, themes can provide 'outcome propositions' based on the combinations of underpinning subthemes generated from the data (Strauss, 1987, p. 10). Codes and themes are generated to make inferences about the findings evident within the data.

Aided by discussions with my PhD supervisors and support from colleagues who had extensive thematic analysis experience, the codes and themes were interrogated with multiple data sources throughout all phases of the project, which contributed to the credibility of my interpretations. As an iterative process of DBR, the codebook was reassessed through each phase of the study, and data were recoded as required to ensure rigorous analysis was undertaken (DeCuir-Gunby et al., 2011; Fereday & Muir-Cochrane, 2006).

NVivo Pro 12 (QSR International, 2021) was used to organise the codes and observations from the participating classroom teachers and my own reflective journals. The work samples from the children were coded separately, using post-it notes and systematic notetaking using the codebook. The codebook was developed in a hierarchical format to organise the broad code categories and individual code(s). For example, fraction as measure, fraction as an operator and fraction as a relation meanings were organised as broader code categories under rational number knowledge. The underpinning fraction ideas for each meaning served as the fine-grain codes (DeCuir-Gunby et al., 2011). A small selection of the codebook demonstrating the hierarchical format is presented in Figure 3.7.

Figure 3.7

Excerpts of Code Hierarchy Generated in NVivo Pro 12



Note. NVivo Pro 12 alphabetically organises code categories and codes.

As the project progressed, additional codes became evident, such as spatial structuring (Battista et al., 1998; Mulligan & Mitchelmore, 2009) in Phase Two. In conjunction with a review of the literature and discussions with my supervisors and mentors, new codes were developed and applied to the codebook. All data (including that collected before these codes were identified) were reanalysed with the additional codes. The addition of these codes illustrates again the iterative nature of the thematic analysis process, enabled by the DBR methodology.

Becoming familiar with the data is essential in the thematic analysis process. This involves reading and re-reading the data, and in doing so, identifying initial analytic observations about the data and potential avenues for exploration (Clarke & Braun, 2013). This process involved looking for commonalities between children's strategy use (such as spatial reasoning) and their understanding of fraction ideas (within their explanations, work samples and observed behaviours). Identifying these patterns in children's behaviour provided the basis for discussing to what extent and in what ways children developed their understanding of fraction and whole number ideas in this study, informing the development of the local instruction theory.

3.8 Trustworthiness of Findings

Establishing trustworthiness in interpretivist studies requires the researcher to address four main criteria (Anney, 2014; Guba, 1981): credibility, triangulation, transferability, and dependability. The researcher can claim trustworthiness of the findings when the study has met these four criteria (Trochim, 2006). These are each addressed below.

3.8.1 Credibility

Credibility is defined as the confidence placed in the truth of the research findings (Anney, 2014). To address the credibility of the interpretation of findings, both the researcher and classroom teachers acted as co-researchers in the study by collecting individual field notes to compare insights to each lesson. Peer debriefing (Bitsch, 2005) was used to talk through the notes recorded by the researcher and the participating teachers. This strategy helped to establish confirmability where 'establishing that data and interpretations of the findings are not figments of the inquirer's imagination but are clearly derived from the data' (Tobin & Begley, 2004, p. 392) is key. This technique helped to obtain and verify how the researcher was interpreting the children's interactions in each lesson, considering how the teachers perceived the learning and children's development in each lesson.

3.8.2 Triangulation

Triangulation is vital for establishing the reliability of findings, also described as the truth value concern (Anney, 2014; Guba, 1981). This process involves using multiple and different methods, sources, researchers, and theoretical perspectives to obtain corroborating evidence (Onwuegbuzie & Leech, 2007). In addition, triangulation helps the investigator reduce bias and cross-examines the integrity of participants' responses. Triangulation involves using a range of research methods (Denzin & Lincoln, 2005; Lincoln & Guba, 1985;). As this project used children's work samples, pre- and post-TBI assessments, observations via research and teacher reflective journals, and daily debrief discussions with the classroom teachers, it is considered to have employed a suitably wide range of methods.

Triangulation of the research team allows different perceptions to become part of the inquiry and helps strengthen the integrity of the findings (Anney, 2014). In this study, the participating classroom teachers also acted as researchers. Moreover, my PhD supervisors and other research mentors also examined different data sources and artifacts to interrogate, challenge and aid me in refining my interpretations of the context.

Finally, triangulation was considered from a participant perspective to enhance the data quality. Children from three separate school settings in Years 1 and 2 were recruited to determine whether the findings were replicated across the same age group, regardless of the school and class environments.

3.8.3 Transferability

Transferability is addressed through the methodological affordance of DBR, where iterative analysis and reanalysis of the data occur. The data are then compared with existing literature, and the thick descriptions provided mean the results apply to other educational contexts and settings. Although this study had a modest sample size, the detailed account of the implementation of the intervention with each class provides findings generalisable to a broader population of early year children. The themes and relationships in the data were rigorously and interactively compared to literature using a transdisciplinary lens, as discussed in Chapter Two. This analysis helped to determine the validity of the theoretical conjectures and determined how generalisable the findings are in addressing the educational problem.

3.8.4 Dependability

According to Bitsch (2005), dependability refers to ‘the stability of findings over time’ (p. 86). Dependability involves participants evaluating the findings and the interpretation and recommendations of the study to ensure they are all supported by the data received from the study informants (Cohen et al., 2007; Tobin & Begley, 2004). It is essentially auditing the study’s analytical methods and procedures and interrogating the findings to ensure they are consistent. This process is a crucial feature of thematic analysis, where a codebook acts as an audit trail for analysis. The methodology of DBR and the inductive component of thematic analysis insist that data is consciously re-examined and recoded for authenticity and consistency. This interrogation occurred several times throughout the study, notably when a new code or theme emerged, instigating the re-coding of previously analysed data. Consulting with my supervisors and peers during this process meant multiple analyses were conducted on the same pieces of data to verify the proposed themes.

3.9 Ethics Considerations

This study was granted ethics approval by RMIT (reference number: CHEAN A 21162-10/17; see Appendix D) and the Government of South Australia Department for Education (reference number: 2018-0013; see Appendix E).

3.9.1 Participant Consent

Participation in this project was voluntary. Written consent was obtained from each school on behalf of the students, the teachers in the project, and the parents or guardians of the children involved. Gaining participants' consent was more than just a process required by formal ethics bodies; it involved mutual agreement concerning what data were collected, how data were collected (in terms of the tasks to be designed, modified, and implemented) and how data would be used in the analysis.

All data collected and published for this study are deidentified. Pseudonyms were assigned to each of the children and participating teachers. The teachers were assigned the label of Teacher A, B or C, consistent with participating Classes A, B and C. The 70 children were assigned the code of Child 1, Child 2, … Child 70 throughout the study. The assignment of codes protects each participant's identity in all published materials.

The participant information sheet and consent forms (see Appendices F and G) detailed the potential risks to participants. The information included the opportunity for participants (and the parent/guardian for children) to ask questions and clarify any aspect of involvement before, during and after consenting to participate in this project, as described in the National Statement on Ethical Conduct in Human Research (2007, updated 2018). All participants in this project provided ‘extended consent’. The data collected in this study can be accessed and used in future and related studies. Sections 2.2 and 2.3 of the National Statement on Ethical Conduct in Human Research informed all sections of the ethics application in this regard.

3.9.2 Data Storage

Data storage and protection is a vital element to plan for and detailed in DBR. RMIT’s 13-step research data management process was studied in detail to consider and plan data collection and storage. I used RMIT’s AARNet cloud storage, which is secure, reliable, and accessible remotely for a large quantity of data. The links provided in this 13-step process, such as the Library’s Research Data Management guide, were used to manage and inform data storage throughout the different stages of the project. All data sources (see Section 3.6) were digitised and stored securely in password-protected files on the AARNet servers.

3.10 Chapter Summary

This chapter has provided a detailed plan for this research project. DBR can be viewed as an interpretivist methodology for exploring educational phenomena; this supports the investigation of a local instruction theory that explains how young children develop fraction ideas through a spatial reasoning approach.

The key methodological choices were intended to examine the conjectured local instruction theory in the following ways. First, the pre- and post-task-based assessment would provide insights into the initial starting point for children’s fraction and spatial reasoning capabilities and their whole number knowledge more generally through items that assess their subitising abilities, number line knowledge, place value understanding and part-part-whole knowledge. The post-intervention assessment data would provide the opportunity to examine children’s (potential) quantitative gains attributable to the intervention and their (potential) qualitative improvements in strategy choice and reasoning capabilities to better understand the influence the instruction sequence played.

Second, the development of an intervention program that provides clear examples of how to construct and structure lessons that include both whole number and fraction ideas with a spatialised approach aimed to provide teachers with what Simon (1995) and Gravemeijer & Van Eerde, 2009) discuss as a pedagogical travel plan. That is, the sequence of learning developed, tested, and evaluated on the basis of a local instruction theory throughout the teaching experiment can be used as a reference for teachers designing their own children's mathematical journeys, based on their individual class's knowledge, skills, and experience.

The next chapter will conclude Phase One of this study with the trial and analysis of the discussed tasks. The construction of the intervention program and refinement of the local instruction theory will be presented and justified.

Chapter 4: Pilot Trial Insights – Class A

4.1 Chapter Overview

This chapter reports on Phase One of the DBR project, the purpose of which was to conduct the pilot trial of the intervention tasks to confirm the sequence and suitability of the conjectured local instruction theory. Section 4.2 sets the scene of this pilot by providing an overview of the participating class and how the trial was organised. Section 4.3 discusses several tasks and how they were analysed for their suitability and inclusion in the intervention program. Section 4.4 presents the refined local instruction theory based on the task analysis and a rationale for the organisation of the intervention program for the teaching experiment. Section 4.5 illustrates two additional representations—spatial language and gesture—that were dominant throughout this trial. The purpose of the trial was to acknowledge new findings and ways children are representing their understanding, so this section discusses the effects of these representations and how this informed the analysis of the teaching experiment. Section 4.6 summarises the chapter and looks towards the first iteration of the teaching experiment in Phase Two, which is the basis of Chapter Five.

4.2 Setting the Scene: Class A

As described in Chapter Three, the participants for this pilot were 26, Year 2 children from a regional South Australian public primary school. Consistent with the requirements for participating in this research, fractions had not been taught to this class in the same school year prior to this pilot commencing. Further, this pilot took place at the beginning of the school year (February 2019), so it was expected that the children had little to no experience with the extended range of fraction ideas this intervention explored and that it may have been some time since they had engaged with any formal fraction instruction.

4.2.1 Understanding the Classroom Environment

After discussing the school's approach to mathematics with the classroom teacher, it was established the children had experienced a two- to three-week unit on fractions in Term 3 of the previous year (approximately September 2018). This unit was aligned to the Year 1 Australian Curriculum content descriptor (version 8.4): *Recognise and describe one-half as one of two equal parts of a whole* (ACMNA016; ACARA, n.d). The unit involved exploring halves of common 2D shapes (pictorial, printed representations), making shapes out of plasticine and colouring shapes/objects in halves and quarters.

This suggested the children had some experience with fair shares in area and continuous models, primarily in part-whole contexts. They may likely be unfamiliar with set or discrete fraction models and fraction meanings such as operator and ratio. It is unknown what spatial reasoning constructs children had previously engaged with or how they had engaged with them during their fraction instruction. Given the innovative nature of the intervention program designed for this study, it was expected that the children may need some explicit scaffolding or sustained exploration of the task contexts.

4.2.2 Organisation of the Pilot

I worked with groups of four to six children at a time to trial each of the tasks. This enabled close observation of how the children were engaging with the activities and the types of strategies and representations they were applying. I worked with each group for approximately one hour at a time, in a separate learning area to their classroom to avoid distractions. Over the course of three weeks, each child participated in multiple sessions and trialled between seven and nine tasks each of the available suite of 22 tasks (see Appendix A).

The conjectured local instruction theory introduced in Chapter Three, constructed from the literature review, is restated in Table 4.1.

Table 4.1

The Conjectured Local Instruction Theory (Version One)

Key indicators of fraction understanding	Characteristics of tasks		Supporting literature
	Primary Fraction Foci	Spatial Reasoning Approach	
Establishing equal parts of collections of discrete items	**Fraction as an Operator:** Fair share Doubling/halving Partitive division/ recursive multiplication **Fraction as a Measure:** Many-as-one	Visual perception of equal groups (drawing on subitising). Recognising relationship between creating shares and recreating the whole from its parts.	Confrey et al. (2014b); Matthews and Ziols (2019); NRC (2006)
Establishing equal parts of continuous items	**Fraction as an Operator:** Fair share Doubling/halving Partitive division/ recursive multiplication Equi-partitioning a single whole Geometric symmetries Similarity **Fraction as a Measure:** Measure Composite unit Unit fraction	A focus on concepts of space for geometric parts—shape, orientation, symmetry in continuous wholes. Visualising the relationship between the shape and size of parts created in relation to the whole.	Confrey et al. (2014b) Bruce et al. (2013); Möhring et al. (2015)

Reinitialising the unit	**Fraction as a Measure:** Unit and composite fraction Equivalent fractions **Fraction as an Operator:** Doubling/ halving Times as many 1-nth-of... …	Visualising measures between parts and wholes, composite and unit fractions, equivalent units. Visualising magnitude relations between parts the distribution of parts.	Bruce et al. (2013); Clements and Sarama (2014; 2017/2019); Confrey et al. (2014b); Confrey and Smith (1995); Siemon et al. (2017)
Splitting as a mental act	**Fraction as an Operator:** Partitive division/ recursive multiplication Times as many 1-nth-of...… **Fraction as a Relation:** Many-to-one Distribution	Visualising the relationship of partitive division/ recursive multiplication, times as many. Stretching/shrinking geometric wholes.	Behr et al. (1983); Confrey et al. (2014b); Lamon (1999)
Connecting multiplicative relations	**Fraction as an Operator:** Partitive division/ recursive multiplication **Fraction as a Measure:** Part-whole fractions Equivalent fractions **Fraction as a Relation:** Distribution Proto-ratio	Scaling and proportional reasoning to determine equivalent units and proportions. Visual awareness of the relationship between part-part and part-whole quantities.	Bruce et al. (2013); Confrey et al. (2014b); Möhring et al. (2015); Noelting (1980); Siemon et al. (2017)

The tasks designed for each key indicator were trialled and the children's responses analysed to determine the tasks' suitability and whether the key indicators were an authentic framework for young children's fraction development. The analysis of the children's responses to the tasks is presented in the next section.

4.3 Evaluating the Tasks

To determine the adequacy and sufficiency of each task for inclusion in the intervention program, the children's responses were analysed using Posner et al.'s (1982) conceptual change framework (see Chapter Three, Section 3.7.1), to determine the extent to which children found the tasks *intelligible*, *fruitful,* and *plausible.*

4.3.1 Task Analysis: Determining Intelligibility, Fruitfulness and Plausibility

While the analysis was completed on all 22 tasks during this pilot, the following examples discussed are demonstrative and are used later in this chapter to evaluate the extent to which the intervention program supports the conjectured local instruction theory.

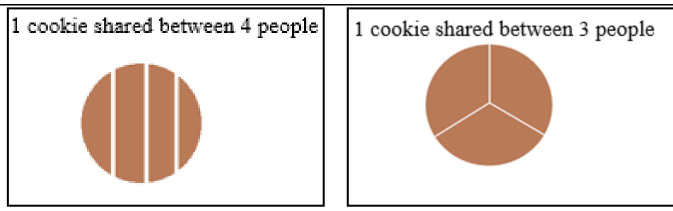
4.3.1.1 Pilot Tasks One and Two

Across the first three pilot tasks, common insights were observed when analysing the suitability of the tasks. Table 4.2 briefly outlines the focus of each task.

Table 4.2

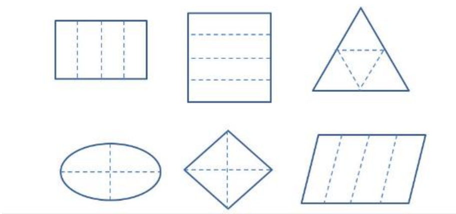
A Summary of Pilot Tasks One, Two and Three

Pilot Task One: Introduce the picture book *The Doorbell Rang* by Pat Hutchins. Children are asked to model and draw how they would share 12 cookies between two, four and six children, and then 16 cookies between two children.
Pilot Task Two: **Task 2A:** *Does each person get a fair share of cookies, in each of the following examples? How do you know?*



Task 2B: Ask students to use counters as a scaffold.
How many ways can you share 12 cookies fairly? What about 16?

Task 2C: Provide children with cards of the following images:



What is different about each shape, and what is the same? (Focus on proportional relationships)
What do these shapes and their parts have to do with fractions? Are there other ways these shapes can be shared fairly? Children may choose to draw representations to describe their thinking.
Refer back to the non-example above and ask the children to explain what is the same and different, how the parts relate to fractions and discuss in relation to the idea of fair share.

Pilot Task Three:
Imagine what ONE cookie would look like if we had to share between two, then four then eight people? What about a rectangular lemon slice?

The tasks were designed with a focus on explaining how fair shares could be created with different shapes and objects, as indicated in Table 4.2. Both discrete collections and singular continuous models were discussed. The majority of children who trialled the task could accept that partitioning or sharing the same collection between different groups of people meant there would be a different number of shares and those shares would differ in size; that is, they recognised the relationship between the number of shares affected the size of each share. However, this did not extend to recognising the relationship between the number of shares and the name of each share. This was evident in Pilot Task One, when asked to share 16 cookies (using counters as a representation) between two people. Several children in the first two groups trialling this task typically placed the counters in groups of two (creating eight shares of two cookies) rather than in two equal groups (of eight cookies). Children 4 and 7, who answered correctly (i.e., two groups of eight), could describe the relationship between the unit fractions created in different collections (e.g., 8 is 1-half of 16; six is 1-half of 12) without being distracted by the number of cookies that created the unit. However, many children became distracted or confused by the number parts created and how many discrete objects created that unit fraction. This suggested a lack of understanding and experience with describing quantities in this way. This point is elaborated on in the next section.

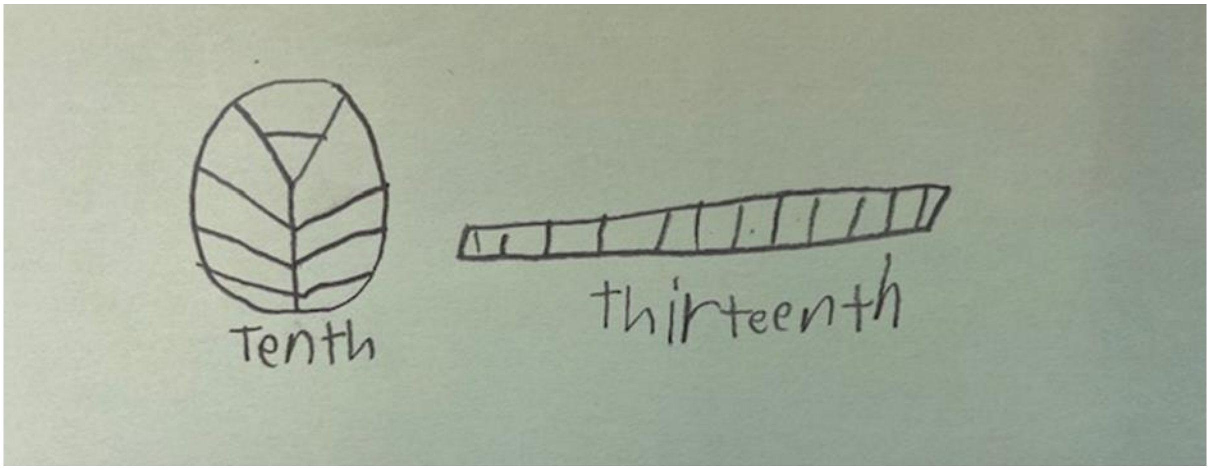
Although the children's interactions with this task suggest a lack of experience with the associated fraction ideas, there was some emerging knowledge evident in nine of the 10 children who trialled the task. The following examples highlight the emerging understandings.

The work samples and explanations that captured this difficulty are now presented for discussion. In Pilot Task 2B, when asked to visualise if and how each of the 2D shapes could be shared fairly in other ways, many children did not seem to be interested in exploring this strategy; rather, they indicated a trial-and-error approach by drawing their ideas. Some children even stated they could not 'see' or visualise the outcome without drawing, suggesting visualisation alone at this point was not a fruitful strategy.

Child 12 created the representation shown in Figure 4.1 and described below.

Figure 4.1

Representation Created by Child 12



> Child 12: There are tenths in this [pointing to the oval representation on the left] and there are thirteenths [pointing to rectangular shaped image]. I just counted the parts as I drew them in.

The representations by Child 12 suggests there is some regard for the equality of the parts, but a focus on the count of individual parts, rather than the relationship between the parts and the whole. Despite this, the child still acknowledged that a whole is partitionable and that there is some regularity to the parts created, demonstrating they find this idea of dividing different wholes intelligible.

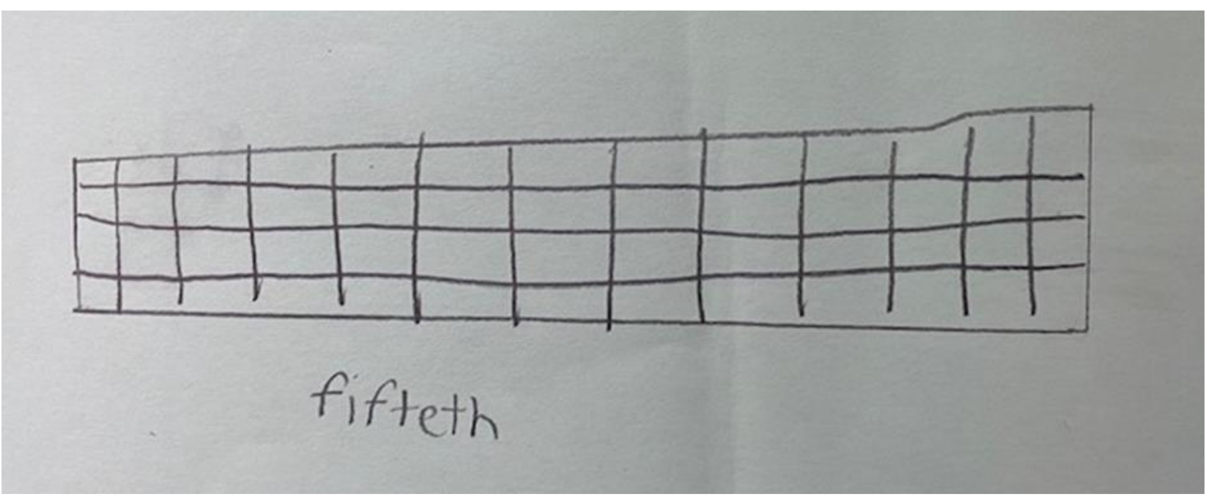
Supporting this analysis was the observation of how Child 12 created the representation. In the oval representation of tenths in Figure 4.1, the child started with an initial mid-line partition from the bottom of the shape until approximately 2-thirds of the way up. Child 12 then started drawing lines from the bottom of the circular shape (in a 'V'-like shape) until they reached the top of the first vertical line. They counted each part they created, and initially recognised they had created eight parts, so they drew a horizontal line through the top segment to create 10 parts in total. The central 'Y' structure of the circular model suggests this child was drawing on their previous experience and familiarity with this model (possibly the common image of a circle partitioned into thirds) and assumed that to create tenths, they would need to base their additional partitions around this central ‘Y’ framework.

Similarly, Child 12 drew the outline of the rectangular shape and started partitioning it from left to right, seemingly ignoring the size and equality of each part. When they ran out of room to continue their partitions, they counted each segment individually and recorded them as ‘thirteenths’. While this child believed they had created tenths and thirteenths, their explanation suggested this was the product of making an iterative and an unintentional number of lines, rather than using spatial proportional reasoning to estimate a deliberate number of predetermined units—even when a benchmark or familiar representation (such as the circle partitioned into thirds) was inferred.

In the next example from Task 2B, Child 10 demonstrated some similarities to Child 12’s thinking in their representation of partitioning a rectangle (see Figure 4.2).

Figure 4.2

Representation Created by Child 10



Child 10: The lemon slice is all cut up, to like, fifteenths, I think.

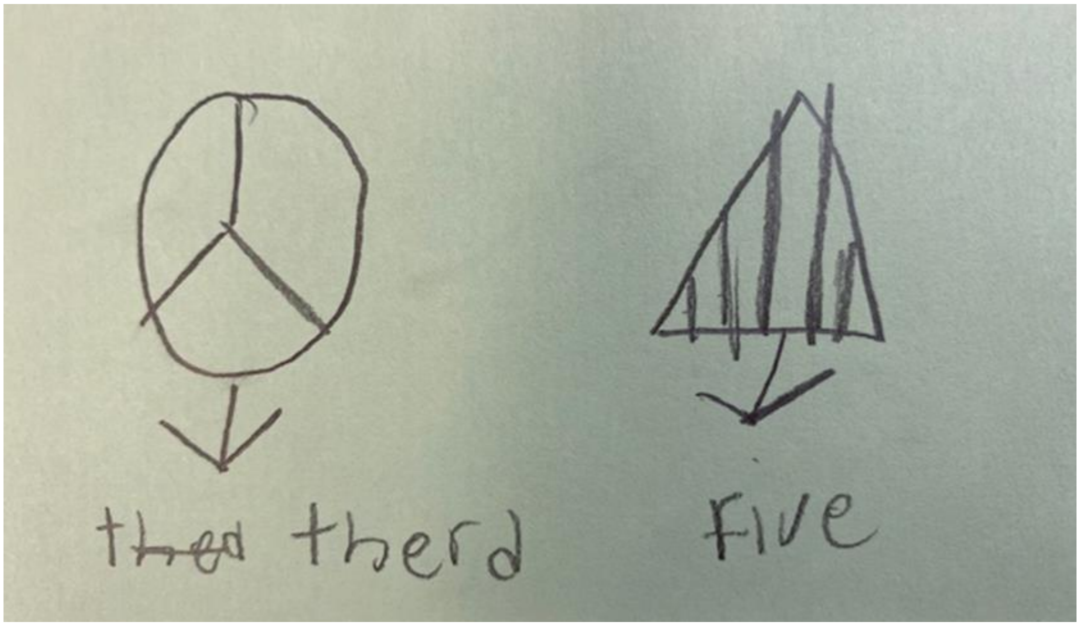
Child 10’s representation could be interpreted as having more regularity in the parts created than Child 12’s representation above (Figure 4.1); however, there is a misunderstanding about the number of parts created and the name given to those parts. Like the previous example, Child 10’s representation does indicate that even though using spatial visualisation or spatial proportional reasoning as a partitioning strategy may not have been fruitful, the child understands that different wholes can be partitioned and, therefore, demonstrates the task is intelligible. However, similar to the representation created by Child 12 above (see Figure 4.1), a multiplicative foundation of partitioning is not evident in Child 10’s representation. Child 10 was observed simply drawing lines vertically from left to right, and then horizontally from top to bottom, rather than halving the region or using spatial proportional reasoning to try and achieve equal units. Plausibility was also not evident in this child’s response with regard to their strategy choice as, when questioned about how they knew they had created *fifteenths*, they responded, ‘I’m not really sure’.

The children’s representations in Figures 4.1 and 4.2 revealed that these two children have a wider vocabulary of fractional terms than was evident in other children’s discussions. Using the plural terms tenths, thirteenths, and fifteenths (although not always accurately) indicated that some of the children had some awareness that the number of parts names the size of the parts—a critical idea for understanding partitioning.

The next example, from Task 2B, also demonstrates the familiarity with area models that many children demonstrated in the tasks. Child 13’s explanation accompanies Figure 4.3, which was used to identify how the child constructed the representation.

Figure 4.3

Representation Created by Child 13



Child 13: I’ve seen this picture before in my classroom [pointing to the image of thirds]. Five cuts gets you five shares [referring to the image of the triangle].

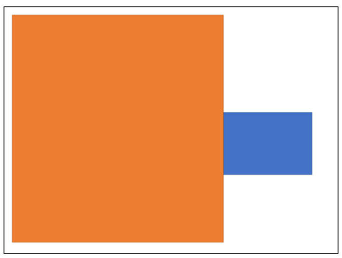
This child’s response indicates that they have an emerging understanding of the relationship between the number of parts and the name of the parts, even though their description of the partitions and number of parts was incorrect. A consistent understanding between the equality of parts in the different shapes was not evident, and, again, the child was observed partitioning the triangle from left to right, indicating an iterating, or counting approach rather than considering how they needed to partitioning the shape to achieve equal parts. However, the child still found the task intelligible, demonstrated by their willingness to partition such wholes, and it is quite plausible that the number of partitions or lines drawn can represent the number of parts—although that is only true for the circular model above.

4.3.1.2 Pilot Task 11: Hidden Fractions

The next example of the tasks analysis is from Pilot Task 11: Hidden Fractions. Table 4.3 outlines the focus of this task.

Table 4.3

Pilot Task 11 Focus

Pilot Task 11: Hidden Fractions
Problem Context: *Part of the blue rectangle is hidden under the orange shape.*  *What fraction of the blue rectangle could be hiding? Explain your thinking.*

The children were presented with the image in Table 4.3, printed on an A4 sheet of paper, and told that it was a picture of a blue rectangle that was partly hidden underneath an orange square. The children were asked to consider the possible size of the blue rectangle and what fraction of it *could* be hiding. The children were asked to explain their thinking.

This task is deliberately open-ended as it gives children the opportunity to explore their thinking in terms of their previous fraction experiences and apply that knowledge to the current context. Four of the nine children who experienced this task demonstrated intelligibility, as shown in the following explanations they gave in response to this task.

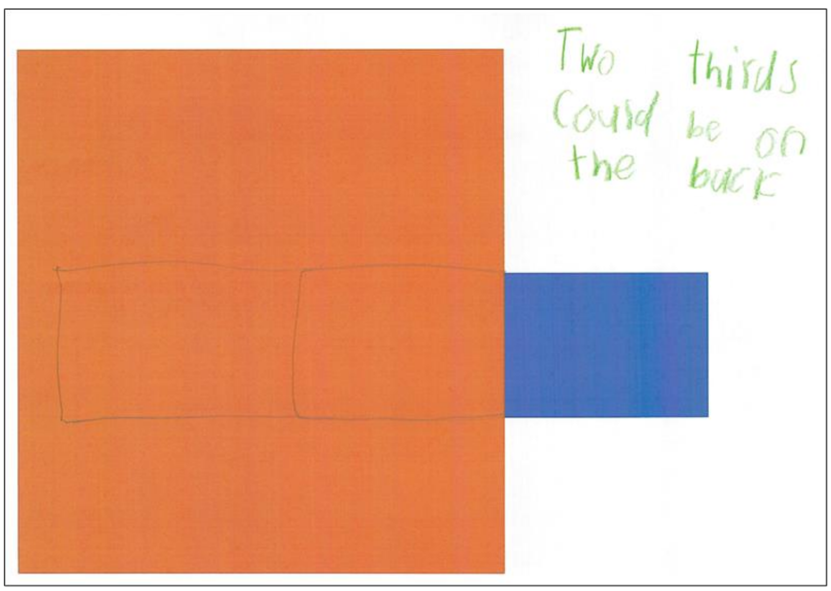
> Child 14: I think maybe a half? If you have this bit [pointed to visible blue section], the same underneath, it makes half and half.

In this case, the child did not draw the missing part (half) that they were explaining; the child used their finger to outline the boundary of the fraction of half on the orange square, suggesting they found this task intelligible in noting the possibility of an unknown fractional part being hidden. Their explanation of the task indicates that they can visualise fractional parts when a whole is unknown, also suggesting this strategy provides a plausible outcome or justification for a half being the hidden fractional part.

Child 22 described a multi-step visualisation process in their explanation: 'If I flipped it [the blue section] over, and then over, it would be three parts. So, thirds, but two are hidden' (see Figure 4.4).

Figure 4.4

Child 22's Work Sample



Child 22 used their hand to indicate how they imagined the iteration of the blue section (flipping their palm over as they described flipping the parts) to create two additional thirds. The multi-step nature of their explanation—flipping 'over and over' with an accompanying gesture—indicates they engaged with spatial visualisation. They also used the word 'same' as they pointed to each third (where the proportions drawn were relatively equal), suggesting they found this context plausible, that is, that 2-thirds of the blue rectangle can in fact 'fit' underneath the orange square. This explanation indicates that via an area model, the child is engaging with the ideas of composite units and possibly the times-as-many ideas necessary for conceptualising fraction as a measure and fraction as an operator meanings (Confrey et al., 2014b). This provides evidence that even with little experience of these ideas, the task is achievable for children of this age.

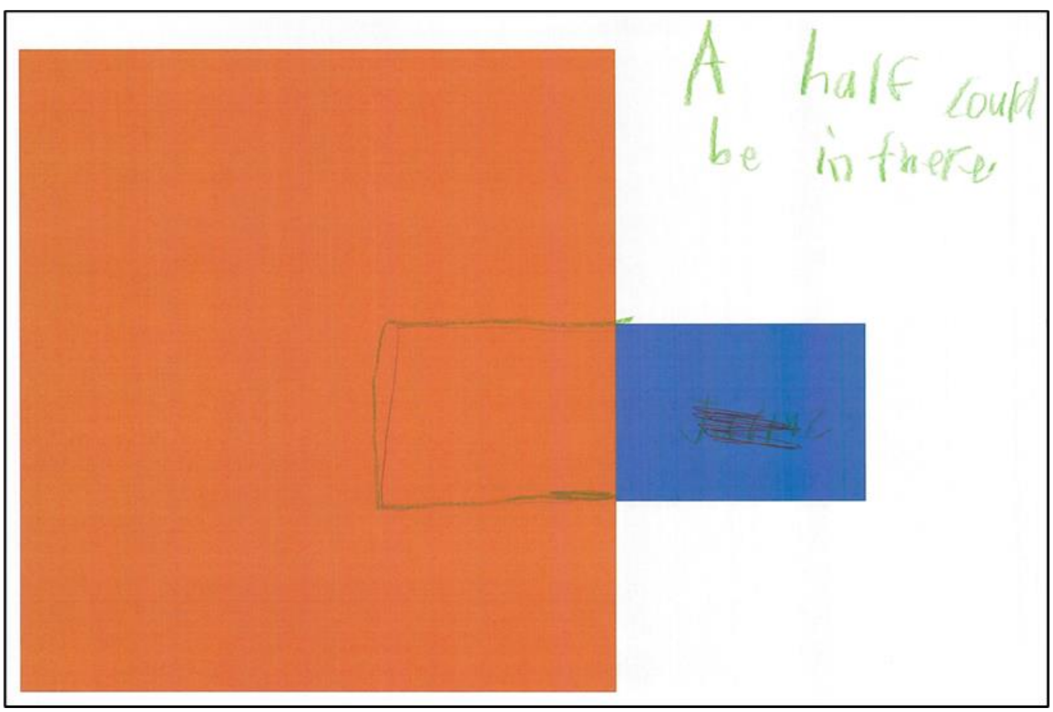
The next example shows the cognitive conflict Child 17 experienced when asked what fraction of the blue rectangle could be hiding.

Child 17: You don't know, 'cos it's hidden [turned the stimulus paper over, suggesting they were checking for clues on the other side of the paper].

After considering the problem for a few moments, Child 17 stated, 'if I used my mind, I could imagine half and half' (see Figure 4.5).

Figure 4.5

Child 17's Work Sample

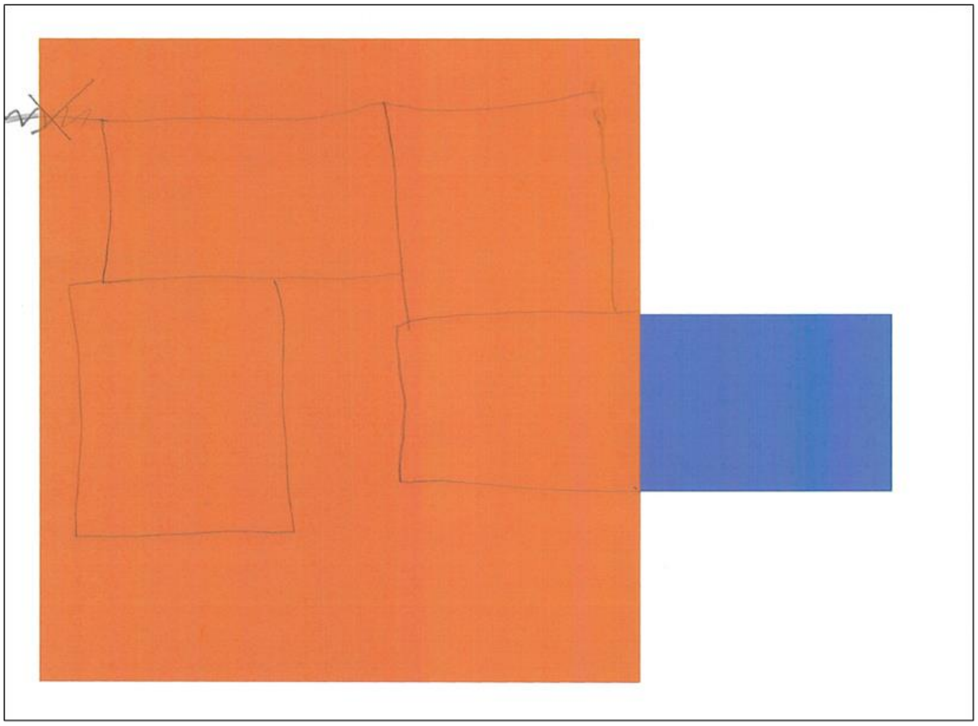


In this representation, Child 17 initially used gesture to describe their explanation of the hidden part by flipping their hand over (from palm down to palm up) after drawing the proposed hidden part. This child spent some time flipping the paper back and forth, appearing to 'search' for the hidden part, before developing an explanation that suggested they were able to visualise a missing part. This behaviour may indicate that the child was not entirely comfortable or familiar with using visualisation as an initial strategy, which suggested a probable lack of experience with utilising spatial reasoning skills, especially when exploring fractions. This also suggested that although the child may have found the task to be intelligible, the strategy of using spatial visualisation was not necessarily deemed fruitful at the beginning of this task. However, the search for meaning and the conclusion that half of the blue rectangle could be hidden within this representation that was proportionally accurate suggests this child believed the context was plausible and that the use of spatial visualisation was, in fact, a fruitful strategy.

The next example demonstrating the intelligibility and fruitfulness of this task is quite different from the previous examples. After looking at the stimulus for a few seconds, Child 16 asked, 'does it have to be in a straight line?' I responded by stating they could represent and explain their thinking in any way they liked. Child 16 proceeded to produce the representation shown in Figure 4.6.

Figure 4.6

Child 16's Work Sample



Child 16: You could have five parts hiding if the paper was shaped like this [see Figure 4.6]. I don't know how to call it a fraction [sic], but there could be other parts like that hidden.

Child 16 demonstrated a complex iteration of the blue unit fraction: an understanding of the equality of the parts as well as the idea that the parts do not need to be connected in regular geometric formation to be considered parts of a whole. However, this thinking and explanation could suggest a misunderstanding of the task in terms of considering the blue region as a single whole, rather than the units within a continuous region or area. Although the child could not name the fractional parts as fifths, they did run their hand in one continuous motion over the five parts and stated 'the strip' of paper, indicating they did find their representation plausible as a continuous whole, just in a different arrangement. This suggested that spatial visualisation was a fruitful strategy in exploring the many-as-one and composite unit ideas for the fraction as a measure meaning.

The above examples demonstrate that children were prepared to consider that there were multiple possibilities to this problem while working in small groups and comparing their answers. Further, fruitfulness was evident in these answers as the emphasis was on visualisation. These children chose to communicate their internal representations and mental process through drawing on the stimulus provided, using gesture, or a combination of both. The use of spatial visualisation was evident in the above examples, demonstrating that children could see a purpose for this spatial ability when solving fraction problems, even if it took some time, such as in Child 17's case.

4.3.1.3 Pilot Task 13: The French Fry Task

Pilot Task 13: The French fry Task was adapted from Tzur's (2019) version of this problem. Table 4.5 summarises the context of the task.

Table 4.5

Tzur's (2019) Modified French Fry Task

Pilot Task 13: The French Fry Task
Mum bought home Maccas for the dinosaurs for tea one night—lucky dinosaurs! But when she got home, they had only put in a small pack of fries to share between everyone! Children will be given different lengths of yellow tape to represent a French fry.  **Task 13A:** *Can you share this fry equally between two dinosaurs?* Attach to the child's partitioning operations (observable through folding—child may fold initially). Tell me about your strategy. Why did you fold the paper into two parts? What is the name of each part you created? How can you convince me that they are halves? **Task 13B***: Share one fry equally among three people. Promote the child's splitting operations through spatial visualisation of parts.* *Questions: Have you achieved thirds? Why/why not? What do you notice about thirds here in relation to halves in your previous task?*

Task 13C: *Share one fry equally among five people.*
Within task questions*: I see you created unequal shares. How can you ensure fair shares? Before you make a guess about the size of the share among five people, look at the size of the shares when we shared among three people. Will sharing between five result in bigger or smaller shares? Describe how you know.*

This task was modified from Tzur's (2019) original task, which anticipated an additive, iterative approach to partitioning. However, the redeveloped version was more complex than anticipated. Of the 10 children who trialled the task, only three appeared to be aware of the relationship between the number of parts and the size of them when considering three and five shares. All of the children were able to fold in half and share the paper fry between two children, but seven of the children across the two different groups appeared to use a trial-and-error strategy to partition into three, four and five parts. They demonstrated this by starting with one end of the 'fry' and iteratively folding the strip over itself, rather than considering the parts in relation to each other and the whole. The children created many more parts than required and justified this by counting the number of parts created and saying that is how many dinosaurs they could feed.

Further, nine of the 10 children who trialled this task wanted to discuss the regularity of the strip of paper used. For example, Child 23 stated, 'a real fry would have slanted ends', referring to how a French fry often has a diagonal 'end', which would affect the equality of the parts. This shows an awareness of the fair share idea in relation to the parts; however, this task counting approach rather than partitioning, possibly due to the difficulty with working with thirds and fifths. The nature of the paper strip in this task seemed to contribute to the trial-and-error approach taken by children, rather than the visualising and partitioning emphasis intended in this task. On this basis, this task was omitted from the intervention program.

4.3.1.4 Pilot Task 20: Bags of Wool

There was only one task, Pilot Task 20: Bags of Wool, for which the nominal quota of two children (from the sample of 10 children that trialled the task) was not initially reached. Table 4.4 outlines the focus of this task.

Table 4.4

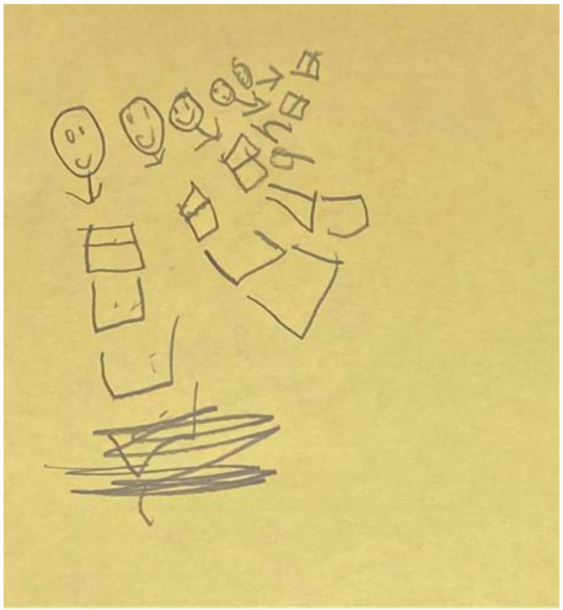
Pilot Task 20 Focus

Pilot Task 20: Bags of Wool
Do you all know the nursery rhyme 'Ba Black Sheep'? Let's sing it together! *If the sheep produced three bags of wool—one for the master, one for the dame, and one for the little boy—how much wool would each person receive if they had to share three bags between five people? Can you estimate/ visualise or draw approximately what each share will be?*

Only one child found this task intelligible and the idea of sharing in this context plausible. Child 5's representation of this problem is presented in Figure 4.7 and discussed.

Figure 4.7

Representation Created by Child 5

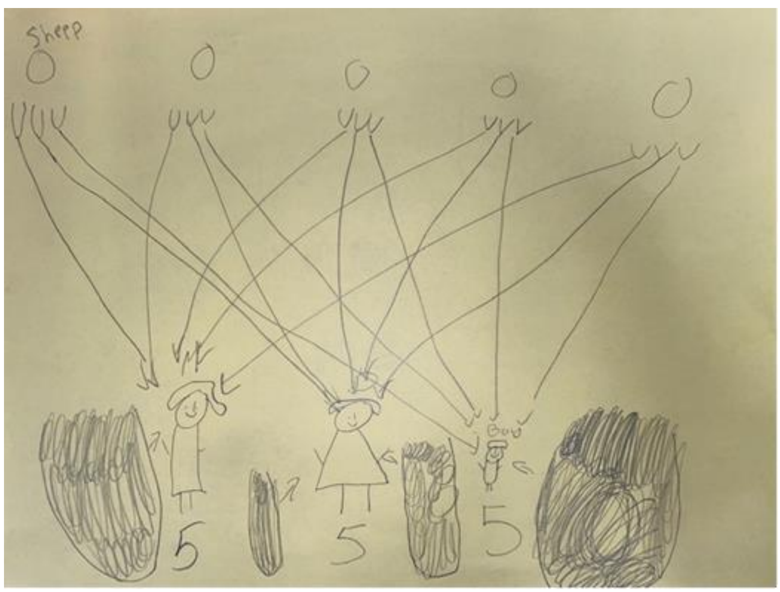


While I was not expecting the children to name the share as *3-fifths* per se, I was examining whether the children could work with such a problem and represent their thinking for this task—which was notably complex for this age group. Although Child 5 demonstrated the understanding that multiple continuous wholes can be shared fairly (illustrated by an attempt to partition some of the bags of wool in what appears to be half), they were not convinced they had a sufficient strategy for completing the task. It is evident in their work sample that Child 5 was confused about the quantities to be shared and by how many, as they have predominately drawn three bags of wool *for each* of the five people, rather than three bags to be shared between five people.

This task was a modified version from Siemon et al. (2017), which in its original form, was the basis of a many-to-one context: *'Each sheep produces three bags of wool. What if there were five sheep—how many bags of wool?'* After observing children's difficulties with my version, it was reintroduced to the children as the original. The original version provided a basis to develop children's multiplicative knowledge by asking children to consider and name the quantities using different referent units (Wilson et al., 2012). That is, naming three bags per person (many-to-one idea) supports early ratio understanding, while naming one share as three bags of wool (many-as-one) promotes an understanding of rational number as a quantity that can be measured in relation to a whole (Wilson et al., 2012). This task was based on whole number contexts, which is consistent with Confrey et al.'s (2014b) theory for rational number knowledge whereby the contexts for 'dealing, splitting, and distributing multiple wholes to sharers' (p. 725) provides a multiplicative foundation of the target understanding of partitioning and unitising for fractions. This task supports the development of the last key indicator of the conjectured local instruction theory, therefore an additional group of five children piloted the redeveloped version of this problem. The representation created by Child 11 (see Figure 4.8) illustrates that despite wanting to represent how many bags of wool each person would receive (which was outside the requirements of the task), they were able to identify that for five sheep there would be 15 bags of wool in total. Intelligibility was, therefore, demonstrated by identifying the outcome of five sheep producing three bags of wool, as well as this child's understanding that this collection can also be represented as five bags of wool per person—when also incorporating the three characters of the nursery rhyme into their representation.

Figure 4.8

Representation Created by Child 11's of the Redesigned 'Bags of Wool' Problem



Fruitfulness was demonstrated in the multiple components of the representations Child 11 produced. Although Child 11 initially drew each bag of wool and counted individually to determine a total of 15, and then shared each bag one by one to the three characters in the nursery rhyme, recounting each character's collection multiple times to determine they each received five bags each. These early distribution strategies imply Child 11 saw purpose in engaging with this form of representation for exploring division and distribution ideas. Three other children also produced representations that indicated they too found the task intelligible and their representation strategy of distribution fruitful, by drawing the five repeated units of three bags of wool.

4.3.2 Summary of Task Evaluation

Overall, the children found the tasks to be more intelligible than fruitful or plausible across the suite of tasks. This is not surprising in that to consider a concept, strategy, or idea within a task fruitful means they are using their learned knowledge and strategies to solve problems previously unsolvable (Hewson & Thorley, 1989). The children have had little if any experience with solving fraction problems like those presented in this study, and as the children in this cycle were not experiencing the complete intervention program, they did not have the opportunity to explore these ideas in a sustained manner. Despite this context, the pilot provided valuable information about how and when the specific concepts and strategies in each task may be used in the intervention.

On the basis of the conceptual change framework analysis, several tasks were either omitted or modified for the final intervention program (see Table 4.6).

Table 4.6

Summary of Changes Made to Tasks for the Intervention Program

Pilot task number		Children's engagement	Reason for modification or omission
1	Sharing Cookies	The children tended to become distracted by their drawings rather than focus on visualising the outcome of creating equal shares.	This task was modified slightly to include opportunities for children to fold different paper-based regions before drawing, to help support the visualisation and spatial proportional reasoning processes. The focus was on discrete collections only; however, it was modified to include mixed fractions (eight cookies between 12 people; see Appendix C).

8	Who ate more pizza?	Although this task was trialled by the greatest number of children (*n=11*), only four found the idea of equivalence *intelligible* and only three found this idea *fruitful*. Children were also reluctant to visualise parts before representing pictorially, which resulted in a more trial-and-error approach rather than any systematic reasoning of the concepts.	This type of problem would be better suited after partitioning and unitising have been established. Children do have other opportunities to explore equivalence in paper-folding tasks, geometric shape task and other cookie sharing and partitioning tasks. This task was therefore omitted.
11	Hidden Fractions	This task elicited lots of spatial language and three forms of gesture in the children's responses. However, it highlighted that if children do not have experience with or understanding of different representations of fractions, then determining an unknown fractional part is very challenging.	This task is similar in conceptual foci and spatial skills to several other tasks (e.g., Tasks 7, 14, 17). This task only took each group approximately 10 minutes to complete and was not as hands on or exploratory as the other three similar tasks. This task was omitted from the program but kept as a possible warm-up activity.
12	Chocolate Ratios	Three of nine children found this idea of distribution as *intelligible*, and two were able to describe the *fruitfulness* of the concept and describe the ratio of 1:2. Many children were more concerned with cutting up each block so that they 'looked the same', rather than considering the possible ratio of small and large blocks.	This was deemed an important task to include as it challenged the children's ideas about the *measure* and *part-whole* concept to a *for each* idea. It was determined to be best used as a warm-up task rather than the main body of a whole lesson, to introduce children to the ideas of *fraction as a relation*.
13	French Fry Task	Only one-third of the children who trialled this task found it *intelligible*. Many used a guess-and-check approach using iteration rather than splitting to determine the required share.	The Cuisenaire task (Task 21) offered more scaffolding to support this idea for this age group. Therefore, this task was omitted from the intervention program.
14	Finding Wholes	The children did not seem to understand the context of the task as well as expected. The task also promoted a part-whole understanding more than I anticipated, rather than the connection between *times as many*, *1-nth-of...* and *composite units* that I intended.	The task was similar to Pilot Task 17: Animal Proportions, which seemed to evoke a connection between the ideas better and provided a more engaging context for children to explore. On this basis, this task was omitted.
18	Plant Growth Rates	This task drew far more on children's number fact knowledge than it did on their spatial reasoning skills. Half of the children who trialled this task were able to answer correctly almost instantly without the use of materials and or	Although this task did not elicit spatial reasoning skills per se, this task was included in the pre- and post-assessment, rather than the intervention program, for two reasons. The first was to explore children's use of representations and

		representations. Those who could not, did not recognise it as a rate problem, and simple answered that Plant B was growing faster, which indicates absolute rather than proportional thinking.	whether this included a spatial component, and the second was to determine whether children responded differently to the question in the post-test, after exposure to a range of different meanings of fractions and spatial reasoning exposure.
20	Bags of Wool	Only one child attempted a representation of the original problem.	This task was redesigned during the pilot and included in the pre- and post-assessment (see Section 4.3.1.4).
22	Dinosaur versus Human	The task was similar to Pilot Task 17: Animal Proportions; however, the lack of concrete materials made this task more challenging than anticipated. Many children did not necessarily pay attention to the mathematical nature of the problem (the relationship between a human and dinosaur sizes), preferring to spend more time drawing and finessing their drawings of a particular creature.	Given the similarity in conceptual focus this task had with Task 17 and the fact it seemed more challenging and time consuming (given the focus on drawing), this task was omitted.

4.4 Revised Local Instruction Theory

The purpose of trialling the tasks was not only to determine their suitability for the intervention program, but to examine whether the key indicators of the local instruction theory are appropriate learning goals for this age group. Further examination of the children's responses to the suite of tasks in this pilot suggested that some of the key indicators needed to be revised. This was to reflect the capabilities the children demonstrated in the pilot, and to better describe how the learning was anticipated to develop over time in the teaching experiments.

Table 4.7 presents the revised local instruction theory (version two).

Table 4.7

The Revised Local Instruction Theory (Version Two)

Key indicators	Characteristics of tasks	
	Primary Fraction Foci	**Spatial Reasoning Approach**
Creating and justifying equal shares	**Fraction as an Operator:** Fair shares Doubling/ halving 1-nth-of...... Partitive division/ recursive multiplication Geometric symmetries **Fraction as a Measure:** Many-as-one Measure Composite units	Visual perception of equal groups (drawing on subitising). Equality of parts regardless of model (i.e., equal parts for discrete collections and continuous models less than and greater than 1). Visual awareness of structure of parts through geometric regularities—shape, orientation, symmetry. Spatial visualising transformations of parts and shapes.
Reinitialising the unit	**Fraction as a Measure:** Composite unit Unit fractions Equivalent fractions **Fraction as an Operator:** Doubling/halving Partitive division/ recursive multiplication	Visualising multiple ways to create composite and unit fractions through unitising. Visualising magnitude relations between parts and spatial structures for creating and reinitialising discrete sets of units.
Recognising proportional equivalence	**Fraction as a Relation:** Distribution **Fraction as an Operator:** Doubling and Halving Times-as-many 1-nth-of...… Scaling **Fraction as Measure:** Composite units Unit fractions Equivalent fractions	Visualising proportional relationships between fractions created, of same and different wholes.

Connecting multiplicative relations	**Fraction as a Relation:** Many-to-one Distribution Proto-ratio **Fraction as an Operator:** Partitive division/ recursive multiplication Times-as-many 1-nth-of...… **Fraction as a Measure:** Part-whole fractions Equivalent fractions	Visual and structural awareness of number relations between part-part and part-whole quantities.

Each of the key indicators are now discussed and their place in the revised local instruction theory justified.

4.4.1 Key Indicator: Creating and Justifying Equal Shares

To recap from Chapter Three, the design of the tasks was informed by Confrey's (1994) notion of splitting: 'in its most primitive form, splitting can be defined as an action of creating simultaneously multiple versions of an original, an action often represented by a tree diagram' (Confrey, 1994, p. 292). Confrey argued that young children's rational number reasoning (which includes whole number and fraction ideas) develop from recursive acts of splitting that develops in parallel to their 'counting worlds' (Norton & Wilkins, 2013, p. 8). That is, splitting involves paying attention to the magnitude of the splits often through doubling and halving that evoke symmetry, similarity and proportional awareness between the quantities generated, not the counting of parts or collections.

The children demonstrated a competency with both continuous and discrete models; however, the discrete sets typically evoked counting strategies or a tendency to confuse the number of objects and the relationship to the set—most likely due to the children's lack of experiences with the fraction ideas. This lack of experience suggested that for the target age group, the starting point for the local instruction theory is the development of the fair sharing idea and the relationship between continuous and discrete contexts. The first and second key indicators of understanding for this age group was to focus on equal parts of discrete collection and then equal parts for continuous whole, respectively. In the analysis of the pilot tasks, it was clear that children are aware of what fair shares and equal parts mean in the context of both continuous and discrete, but due to their lack of experience with fractions, children tended to rely on their whole number knowledge in the form of counting parts as their primary strategy. Given there is a focus on spatial reasoning in this study, combining the first two key indicators of exploring fair shares simultaneously in discrete and continuous context provides a greater foundation for developing partitioning (based on splitting) through spatial visualisation and comparing the magnitude of the shares created. This key indicator of *creating and justifying equal parts* was the major focus of the first four lessons in the intervention program, described in Table 4.8. A full description of each lesson of the intervention program is in Appendix C.

Table 4.8

Overview of the Lessons Related to Key Indicator: Creating and Justifying Equal Shares

Lesson	Fraction Foci			Spatial Reasoning Approach	
	Fraction as a Relation	**Fraction as an Operator**	**Fraction as a Measure**	**Spatial Construct**	**Context of Lessons**
Lesson 1: Sharing Cookies		Fair shares Doubling/ Halving Partitive division	Many-as-one Measure	Spatial visualisation	Visualising partitive division/recursive multiplication between parts/shares and whole. Conceiving the change in size of share (discrete and continuous) as more shares are required. Visualising sharing outcomes involving mixed numbers.
Lesson 2: What is a fair share?		Fair shares Doubling/ Halving Partitive division/ Recursive multiplication	Unit fractions Composite units	Spatial visualisation	To partition small sets (<20) and continuous models to develop awareness of the size of the parts and number of parts created. To build visual recognition and awareness of the form, pattern, and regularity of many-as-one parts forming a unit measure. To visualise the act of partitioning to create other partitioning (splits) to conceptualise fraction measures.
Lesson 3: Visualising		Partitive division/ Recursive multiplication	Many-as-one Composite units	Spatial proportional reasoning	Exploring proportional relationships between different shapes that have been partitioned into various fractions (i.e.,

the share of a cookie	Geometric symmetries Partitioning a whole Similarity			quarters, halves, thirds). Reasoning how they represent the same fractions in relation to their whole.
Lesson 4: Sharing divisible collections	Fair share Partitive division/ recursive multiplication Times-as-many	Many-as-one Composite units Unit fractions	Spatial visualisation	Visualising the relationship/action of partitive division and recursive multiplication between sets of objects. Naming and renaming the shares to build a connection between times-as-many and many-as-one shares.

4.4.2 Key Indicator: Reinitialising the Unit

The second key indicator for the revised local instruction theory was largely unchanged. This included a focus on describing relations between fraction unit size and the whole and exploring multiple ways of naming a whole as a set of units. *Reinitialising* to the whole is a term used to describe how a repeated action—such as repeated partitioning (e.g., splitting)—preserves the relationship between an increased number of parts and their decreasing size, underpinned by composite unit and recursive multiplication ideas. It is essentially the unitising concept that was described in Chapter Two.

During the lessons associated with this key indicator, the children indicated an early understanding between unit fractions and composite units from the fraction as a measure meaning. They did so in conjunction with visualising how parts and the whole can be operated on to generate different quantities. For instance, in the pattern block activities the children described different unit fraction, composite unit, and equivalent fraction relationships between the shapes (e.g., equilateral triangles as sixths, and a hexagon as '1'). Moreover, the children demonstrated an awareness (with varying levels of accuracy) that the number of parts names the part, such as descriptions of *fifths, thirteenths,* and *tenths* as an example, in the children's pictorial representations. The lessons in Table 4.9 have been sequenced to support this key indicator.

Table 4.9

Overview of the Lessons Related to Key Indicator: Reinitialising the Unit

Lesson	Fraction Foci			Spatial Reasoning Approach	
	Fraction as a Relation	**Fraction as an Operator**	**Fraction as a Measure**	**Spatial Construct**	**Context of Lessons**
Lesson 5: Cookie fraction estimation		Doubling/ halving Times-as-many	Composite units Unit fractions	Spatial visualisation	Visualising the magnitude of each part in relation to a whole and the total quantity created (equal to and greater than one) through composite fractions.
Lesson 6: Tablecloths	Many-to-one Distribution	Partitive Division/ Recursive multiplication Doubling/ halving Geometric symmetries Similarity	Partitioning a whole Many-as-one Composite units Equivalent fractions	Spatial visualisation Spatial proportional reasoning	Exploring the process of multiple mental folding (spatial visualisation) and rotating parts of the tablecloth to determine proportions of colour, comparing regions of incongruent and congruent wholes. Noticing distributions of composite units and comparing part-part and part-whole relationships.
Lesson 7: Pattern block fractions	Distribution	Doubling/ halving Geometric symmetries Similarity Times-as-many Scaling	Partitioning a whole Many-as-one Composite units Equivalent fractions	Spatial visualisation Spatial proportional reasoning	Similar to Lesson 6, however, children will use pattern blocks to manipulate and create fractional parts. A focus on operating on children's pattern block construction and establishing the relational proportions between the representations (e.g., what would it look like if it were only half

		the size? Double the size? Three times? Etc.).

4.4.3 Key Indicator: Recognising Proportional Equivalence

The original third indicator in the conjectured local instruction theory proposed *splitting as a mental act* through visualising the partitive division and recursive multiplication ideas as the intended learning goal. However, the ability to visualise the act of partitioning was critical to the children justifying equal shares in both discrete and continuous contexts, and to understanding how to name and rename fractions in the previous two key indicators. The tasks that were trialled in this pilot relating to mapping contexts revealed that the children were capable of developing an early appreciation of proportional equivalence (albeit with limited exposure), thus the key indicator was revised to reflect this potential.

The children were able to *recognise proportional equivalence* through visualising the process and outcome of partitioning similar and unlike wholes. It builds on from the previous key indicator were children explored the multiple ways in which a whole can be partitioned and named, to describing how different wholes have been partitioned. For example, the associated lessons enabled the children to experiment with comparing fractions of pathways between their pictorial representations of maps and the larger, carpet maps. Specifically, spatial proportional reasoning was a key feature of the tasks designed to explore how to compare different measures that represented the same fraction idea (e.g., half of the pathway on a physical map is equal to half of the pathways on a smaller, scaled version). Table 4.10 describes the focus of these lessons in the intervention program for this key indicator in detail.

Table 4.10

Overview of the Lessons Related to Key Indicator: Recognising Proportional Equivalence

Lesson	Fraction foci			Spatial Reasoning Approach	
	Fraction as a Relation	**Fraction as an Operator**	**Fraction as a Measure**	**Spatial Construct**	**Context of Lessons**
Lessons 8–10: The dinosaurs have escaped (Parts 1, 2 and 3)	Distribution, Proto ratio, Equi-partitioning multiple wholes	Doubling/ halving Scaling Times-as-many Geometric symmetries, Similarity	Composite units Unit fractions Equivalent fractions	Spatial proportional reasoning	Estimating fractional lengths of paths on carpet maps through benchmarking of half/repeated halving. Paths are not straight, so children need to engage in spatial visualisation and mental rotation to create such benchmarks on the length of multiple paths and use spatial proportional reasoning to compare measures within a single pathway/region. Creating scaled representations of fractional paths, naming and describing the distributions of the measures: half of …path is greater than half of another path.

4.4.4 Key Indicator: Connecting Multiplicative Relations

The final key indicator is unchanged from the conjectured local instruction theory first introduced in Chapter Three. The aim for learning here is that children develop an appreciation for the multiplicative relations of fractions and simple ratio. Pilot tasks 12, 16, 18 and 20 all focused on exploring simple ratio ideas, which the majority of the children found quite challenging. A small number of children (n=5) were able to comment on the part-part relationship between the quantities dealt with in each task (such as ratio of chocolate bars, dinosaur, and human steps). Furthermore, some were also able to move flexibly between the many-to-one and many-as-one ideas in the Bags of Wool task, which is complex for this age group. These findings show that it is possible for children 6 – 7 years of age to work with simple ratio and early multiplicative relations. For the intervention program, the focus on 'seeing' and 'appreciating' the connection between fraction and ratio is promoted through Lessons 11–13, where part-part and part-whole quantities are explored simultaneously between continuous and discrete contexts (see Table 4.11).

Table 4.11

Overview of the Lessons Related to Key Indicator: Connecting Multiplicative Relations

Lesson	Fraction foci			Spatial Reasoning Approach	
	Fraction as a Relation	Fraction as an Operator	Fraction as a Measure	Spatial Construct	Context of Lessons
Lesson 11: How many steps?	Many-to-one Distribution Proto-ratio	Doubling/ Halving Partitive division/ recursive multiplication Times-as-many 1-nth-of… Scaling	Composite Units Part whole fractions Equivalent fractions	Spatial visualisation	Children explore discrete part-part relations of footsteps (i.e., dinosaur: human quantities). Focus is on visualising the complexity of the many-to-one relationship in the outset (i.e., *for each* Dino step, we take many steps to travel the <u>same</u> distance).
Lesson 12: Animal Proportions	Distribution Proto-ratio	Doubling / halving Times as Many 1-nth-of...	Composite units Unit fractions Part-whole	Spatial proportional reasoning	Exploring and preserving continuous proportional quantities and relationship when replicated (enlarged/shrunk).
Lesson 13: Feeding Dinosaurs	Many-to-one Distribution Proto-ratio	Partitive division/ recursive multiplication Scaling 1-nth-of...		Spatial visualisation	Children explore discrete part-part relations of pies: dinosaurs.

4.5 Additional Insights into Children's Use of Representations

In addition to examining the intelligibility, plausibility, and fruitfulness of all 22 tasks, I conducted thematic analysis on the pilot data for two purposes. The first was to test the existing codebook I had generated in the study's preparation phase to determine if the codes and code categories were suitable for analysing the data and generating preliminary themes, in the form of examining the relationship between the fraction ideas, representations and spatial constructs proposed in the key indicators. The second was to determine if there were any new insights that emerged during the trial of the tasks that warranted the inclusion of new codes or categories to the codebook to enable all aspects of the of the children's mathematical behaviour during the tasks to be effectively analysed. Two key insights emerged as important factors in children's development and communication of fraction ideas: use of spatial language and use of gesture.

4.5.1 Children's Use of Spatial Language

In the analysis of each of the pilot tasks, it became evident that children were using specific terms that were spatial in nature; that is, children would refer to terms that were dynamic indicating force or movement (Landau, 2017), such as flipping, moving, and turning, as they described creating fractional parts. Other terms indicated reference to the characteristics of objects and the spatial relationships identified between objects and their parts, or between different objects (Landau & Jackendoff, 1993). This included terms like side, back, front, edge, and corner, when children describe parts or characteristics of objects, and beside, next to, under, and between when referring to relationships among parts or objects. Drawing on literature (see Cannon et al., 2007; Casasola et al., 2020; Gentner & Bowerman, 2009), I identified three categories by which to describe these spatial language terms: *spatial transformations, spatial dimensions,* and *spatial prepositions.* Each term will now be defined and examples of how each of these terms infer children's thinking will be provided. In addition, this section will provide examples of how multiple types of spatial language can be used simultaneously in the same activity.

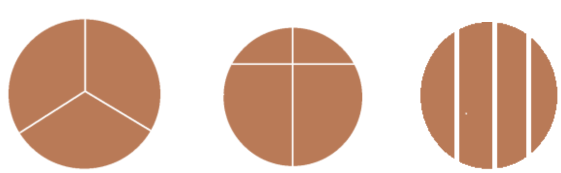
4.5.1.1 Spatial Transformations

Spatial transformations are verbs that describe the movement of an object through space (Chatterjee, 2008). These terms are used to indicate the mental transformations they performed, such as, turn, spin, slide, push, move, rotate, and flip when describing how they determined the correct pairs. In the analysis, it became apparent that those children who were successful at identifying all or at least some of the matching pairs in Pilot Task 4: How much of the cookie did the mouse eat? also demonstrated a greater spatial vocabulary than those who were unsuccessful. This was evidenced by the way the children were able to describe the reasons why some parts could simply not match with others. Children typically described a 'folding' or 'flipping' action to determine the matching pairs or referred to moving a part of the cookie by 'turning', 'spinning' and 'sliding'.

Similarly, in the first part of Pilot Task 2: What is a fair share? the children were asked to consider a range of different sharing contexts (both discrete and continuous models) and explain whether they thought they had been partitioned into fair shares (see Figure 4.9).

Figure 4.9

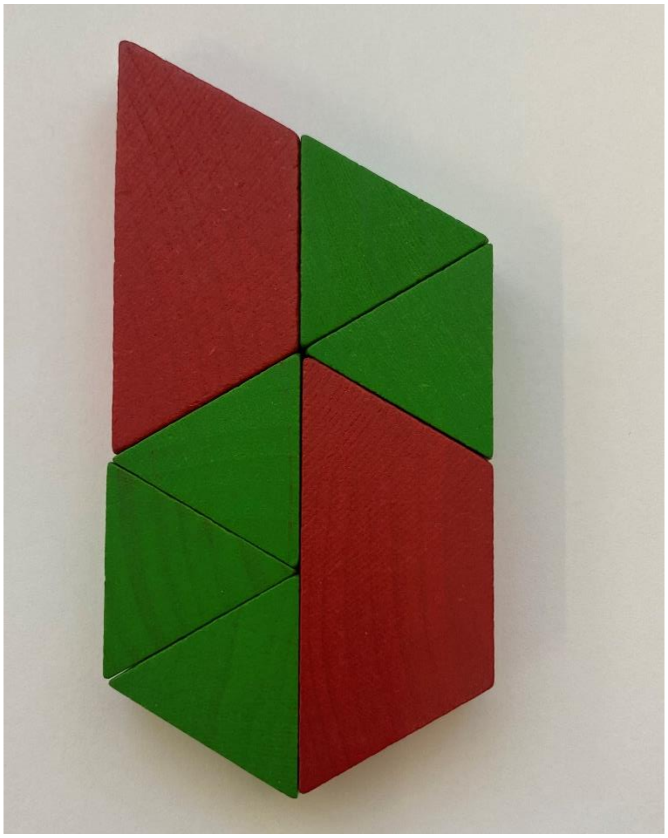
Stimulus Provided for Pilot Task Two: What is a Fair Share?



Seven children described how they imagined 'moving', 'spinning' or 'stacking' parts of the cookie to determine whether the share was fair or not in the continuous examples. Further, Pilot Task 7: Pattern block fractions also elicited children's use of spatial transformations. Here, the children were invited to explore making different fractional representations using the pattern blocks. Child 6's representation of 'quarters' (see Figure 4.10) and explanation provide a typical example of the interaction between the various tools of representation such as concrete material and spatial language in this activity.

Figure 4.10

Recreation of the Pattern Block Representation by Child 6



This model is incomplete if it were to represent quarters as the child intended (i.e., 1-twelfth in the form of a green triangle is absent). Although an incomplete representation, Child 6's explanation referred to using one of the trapeziums to 'fold over sideways' (left) to get two of the quarters and continued to 'fit' the rest of the shapes around this to achieve 4-quarters. The use of spatial transformational vocabulary helped this child explain how they conceptualised quarters in this context, using shapes that could (but did not in this case) create equivalent parts.

In all of these examples, the use of spatial transformation terms helped to determine *how* the child engaged in spatial reasoning constructs. In these examples, spatial visualisation was inferred because the terms describe some kind of movement the children were imagining.

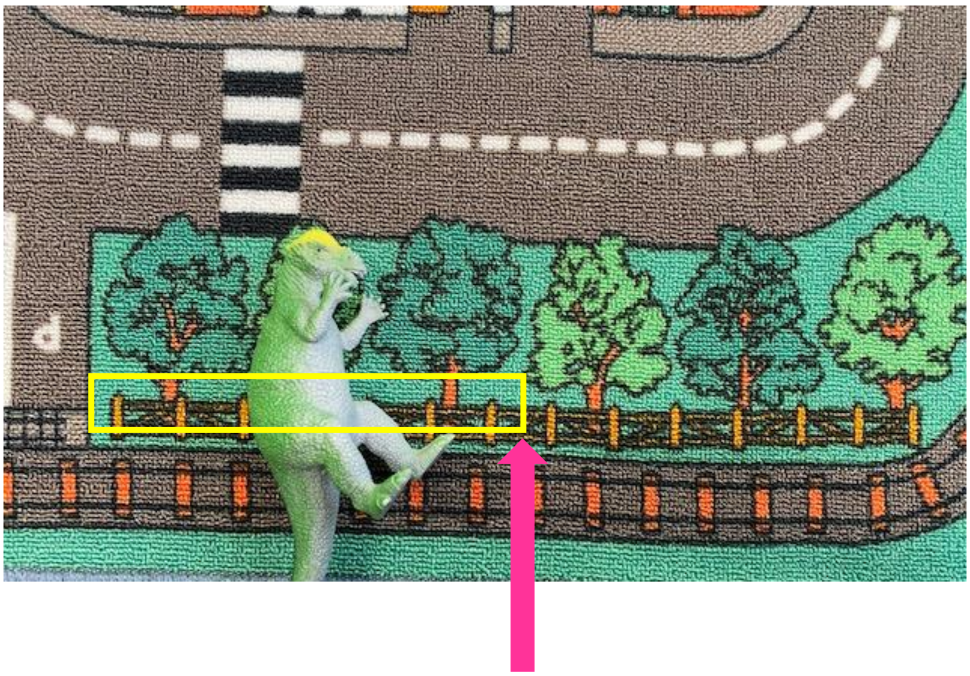
4.5.1.2 Spatial Dimensions

The second category of spatial language I identified was spatial dimensions. These are terms that refer to the geometric properties of an object or space (Casasanto & Bottini, 2013). Terms such as short, big, curve, straight edges, corners, point, wide, narrow, skinny, and fat, were prevalent in the children's responses when reasoning about the size of fractional parts. For example, Pilot Task 13: French Fries (adapted from Tzur, 2019) required children to take strips of paper that represented a French fry to create different numbers of fair shares (e.g., share the French fry between three people). Nine of the 10 children who trialled this task not only referred to the varying lengths of the 'French fry' and its parts but also wanted to discuss the regularity of the strip of paper used. That is, a 'real' French fry would have 'diagonal' or 'slanted' or be described as 'bendy' and 'curved' with reference to its boundary. The relevance of these descriptions is that the children noted the geometric properties of a real French fry would ultimately affect the equality of the parts; therefore, the children's use of spatial dimensions appeared to be connected to their understanding about the equality of the parts in question.

In the next example, several children used spatial dimensional terms to reveal their thinking about a fraction as a measure problem. Task 10: The dinosaurs have escaped! (Part 3) required children to locate the position of each dinosaur on a large carpet mat, based on a set of instructions created by the researcher. For example, one of the instructions read, 'a dinosaur was spotted halfway along the wooden fence near the railway line'. The children not only had to interpret the spatial directions given to decode the carpet mat they were given but also utilise spatial proportional reasoning to determine where halfway was along the nominated fence. Figure 4.11 represents how three children interpreted this task and how they used spatial dimensional terms to explain their thinking.

Figure 4.11

Recreation of Children 15, 19 and 13's Placement of the Dinosaur



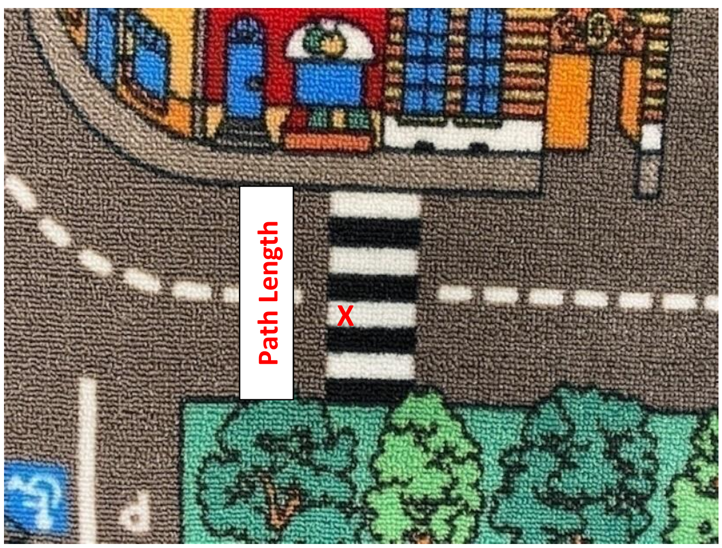
Representation of Child 15's hand placement

Children 15, 19 and 13 all stated that the dinosaur had been placed halfway along the fence line. When I asked the children to explain to me why this dinosaur represents walking halfway along the fence line, Child 15 placed their hand perpendicular to the mat (indicated by the pink arrow) and stated that this point was the middle of the fence, so the dinosaur had to be on one side of this mid-point to have walked halfway. Child 15's description suggests that the child is thinking about the fence as an area model, presumably a rectangle (represented by the yellow rectangle placed around the area in Figure 4.11) and that this portion of the fence is considered a half regarding an area model. The spatial language used in this context suggested it was associated with such thinking.

Spatial dimensions also helped interpret children's understanding and alternative conceptions in other tasks. For example, in the same task, some children could not conceptualise 1-half or 1-quarter of a path. Children 11, 18, 14 and 17 were observed (in different sessions) discussing where a dinosaur would be standing if they were 1-quarter of the way along the pedestrian (zebra) crossing on one of the carpet maps (see Figure 4.12). An 'X' has been placed on the image to represent where the children suggested the dinosaur would be standing.

Figure 4.12

Section of the Carpet Map Used by Children in Pilot Task 10



Note. 'X' represents where the children suggested the dinosaur would be standing.

Although the children were discussing that 1-quarter was the same as a fourth, they proceeded to count four successive black and white stripes and claim this defined area (marked by the 'X') was 1-quarter of the crossing, demonstrating a reliance on the counting of parts and area model representation as described above. What this task required the children to consider was the length of the path as a measure (indicated in Figure 4.12). In addition, the children needed to recognise that the boundary between the middle black and white rectangle was actually 1-half of this measurement, and 1-quarter of this path is the boundary between the first white and the second black stripe (if moving in the direction from the bottom of the picture towards the top). The spatial dimensional terms used by the children in this activity to describe the quarters (incorrectly identified) included terms that focus on four rectangles (also referred to as blocks and stripes by Child 17 and 18). These limitations in the children's understanding of the measurement model were revealed, in part, by their use of spatial dimensional language.

4.5.1.3 Spatial Prepositions

The third language category I identified was spatial prepositions. Spatial prepositional terms are used to describe the location and orientation of an object in relation to other object, or parts of an object in relation to the whole (e.g., next to, on, in, under; Richard-Bollans et al., 2019).

Spatial prepositions were prevalent in eight of the 10 children's descriptions of Pilot Task 4: How much of the cookie did the mouse eat? when determining which fractional parts were paired together. The children who used these terms were far more successful in describing why they thought different parts were paired together. For example, spatial prepositions such as inside, overlap, over the top, on top and between, were used frequently throughout the task to describe what would happen if two circular segments were paired together. Child 16 stated that the third segment, would 'overlap' if they tried to place it in the 3-quarters representation, suggesting this child visualised placing one part on top of another to determine that there would be overlap of the segments and, therefore, a quantity of more than one. Further, this child explained that if they were to place the sixth into the 2-third representation, there would be 'enough room to put another piece *next to* the [sixth]', although they could not name what size the parts were.

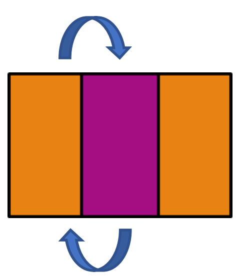
Across this cycle, spatial prepositions were the least prevalent forms of spatial terms captured in the children's responses; however, they were often used in combination with spatial transformation and dimensional terms. This multiple use of spatial language within the same task was evident more generally, as illustrated in the following examples.

4.5.1.4 Children's Interconnected Use of Spatial Language

Pilot Task 9: Tablecloths (see Table 4.9) reveals the use of both spatial transformation and prepositional terms such as referring to moving the parts together. For example, Child 9 describing 'smooshing' parts together, and other children indicated they were mentally 'folding or moving' different parts of their tablecloth 'next to' each other to determine fractional measures and equivalent fractions. Child 8 described how they imagined 'flipping' the orange and purple segments in Figure 4.13, so they could 'see' that even though they are not next to each other on the original cloth, the orange portion still takes up more than half of the tablecloth. The use of the term 'flipping' implies the child engaged in spatial visualisation.

Figure 4.13

Child 8's Reference to 'Flipping' Purple and Orange Segments

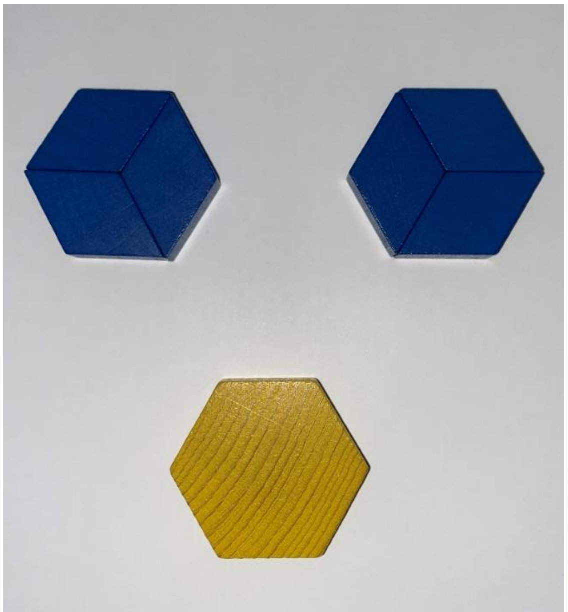


In addition, Child 8 referring to 'more than half' reveals they are also considering the spatial dimensions of this shape to conceptualise the size of the fraction in relation to the whole.

Pilot Task 7: Pattern block fractions also revealed a high level of spatial language in the children's representations of fractions. This task was also open-ended, in that the children were asked to use the pattern blocks to represent everything they know about fractions. Child 7 made the representation shown in Figure 4.14 and stated, ‘the yellow hexagon is half of all the blue shape’.

Figure 4.14

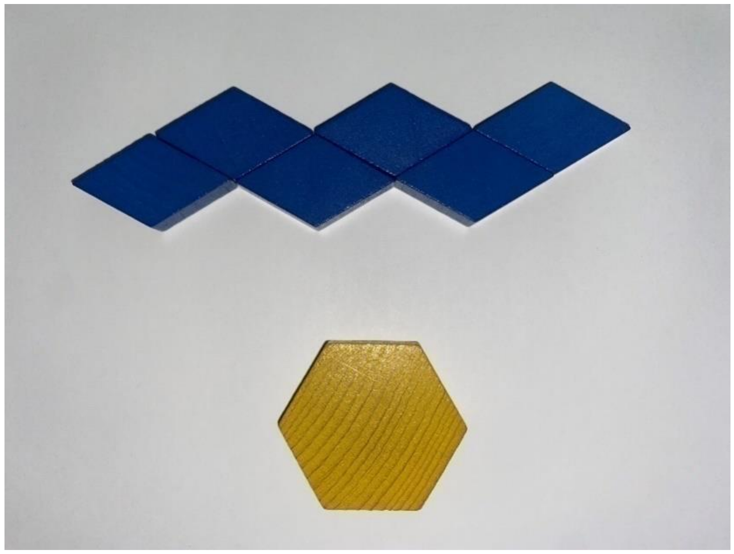
Recreation of Child 7’s Pattern Block Representation of Half



The researcher asked Child 7 if there was another way they could explain the relationship between the shapes (prompting for the doubling/halving idea, equivalence, etc.). Child 7 then rearranged the blocks to the representation shown in Figure 4.15.

Figure 4.15

Recreation of Child 7's Second Pattern Block Model



After considering this new arrangement, Child 7 stated:

> It's still the same, because the parts have just moved around but they still take up the same space. It's just like breaking [the hexagon] into three and doubling it to six pieces. That makes this [hexagon] half of that [blue block collection].

In this description, the child is using spatial transformation terms to explain the proportional relationship between the two groups of blocks. This suggested that the spatial terms help describe the mental transformations they performed when explaining the spatial proportional relationship and emerging multiplicative understandings when referring to doubling of the parts (a fraction as an operator idea). There is also a demonstrated awareness of the spatial dimensions of the individual shapes and how their location and orientation does not affect proportionally the fractional amounts represented (indicating the distribution idea from the fraction as a relation meaning).

Similarly, in the next two examples, Children 15 and 10 stated they could mentally fold half of their model over to replicate the same size unit on the opposite side. Child 15 said they could also ‘cut’ the red part ‘to match the parts in the green’ (see Figure 4.16), indicating an awareness between partitioning in halves and thirds.

This suggests there was a connection between the different spatial transformations described (which is a process of reasoning), an awareness of the spatial dimensions of the regions (which is exploring the size of the shapes used in the representation) the children were visualising (see Figures 4.16 and 4.17).

Figure 4.16

Recreation of Child 15’s Pattern Block Representation

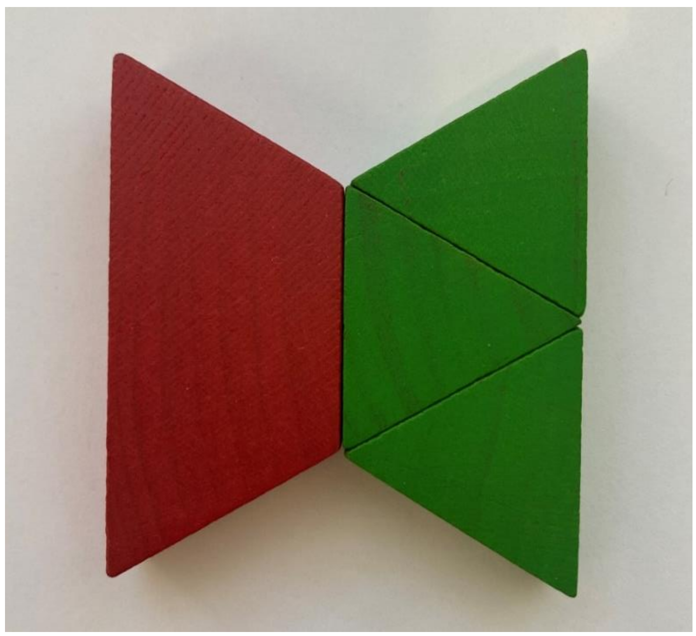
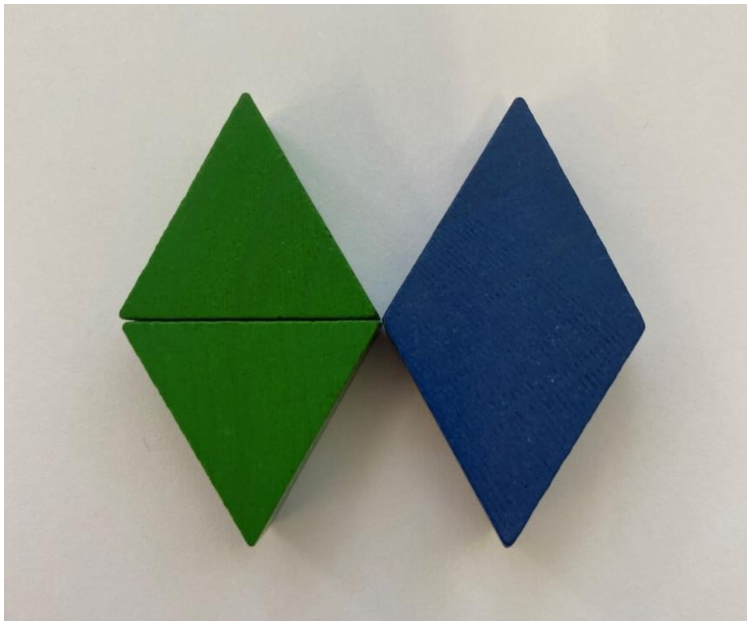


Figure 4.17

Recreation of Child 10's Pattern Block Representation

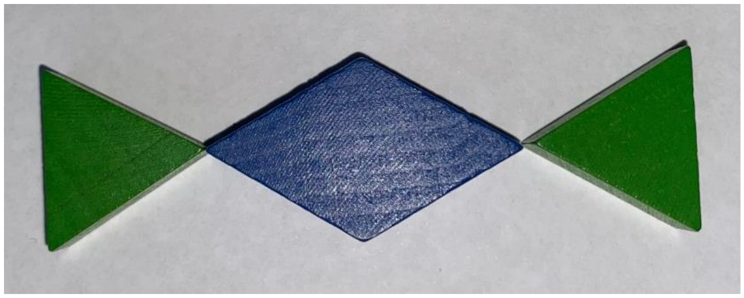


Additionally, both children noted the spatial dimensions of the shapes used in determining the unit size and equality of such parts. This suggested they were focused on the geometric properties of the shapes and able to make direct comparisons about the size of the parts of their nominated fraction. However, it cannot be assumed that these examples indicate that children have a proportional understanding of the parts and the fractions they represent.

This issue was evident when Child 14 joined the discussion with Child 10, Child 15 and myself. I asked the group to consider, 'If the same number of blocks were positioned in a different arrangement, would the fractions we described earlier change?' Child 14 took the blocks from Child 10's above representation (Figure 4.17) and created the formation shown in Figure 4.18.

Figure 4.18

Recreation of Child's 14 Pattern Block Representation



Child 14 claimed that they had thirds because there were three objects, disregarding the size or magnitude of the nominated units. Children 15 and 10 were not convinced of Child 14's claim, although they took some time to provide a counterargument; in this, they stated they 'knew' that the collective green shapes were the same size as the blue area, so even though they are arranged differently, they still represent a total area of half blue and half green.

Analysis of the discussions captured across the pilot, indicated that spatial language was present in some form in children's responses to each of the 22 pilot tasks. These initial findings suggested that while these tasks were not designed with an explicit spatial language focus and that the children in this phase had limited experiences with the intervention program, the children appeared to be intuitively drawn to using spatial dimension terms specifically when exploring the different meaning of fractions. It also suggested that the tasks themselves presented an increased opportunity to explore the spatial dimensions, rather than spatial transformations or spatial prepositions.

These observations suggested that these children may be drawing heavily on their previous geometric or area representations in their initial attempt to make sense of the fractional contexts presented. This premise is supported by their previous experiences of fraction instruction as described by Teacher A, which illustrated a part-whole emphasis with area (geometric)-based models.

The identification of spatial language as a subset of the spoken word representations, provides a finer grain analysis of children's use of language in the intervention. Specifically, it helps determine what and how children are interpreting different concrete /pictorial models and representations in addition to providing further insights into how children may be mentally manipulating different objects to explore the various fraction ideas. Adding the different forms of spatial language identified to the code book provides an additional lens in which to interpret the children's thinking and reasoning throughout the intervention program.

4.5.2 Children's Spontaneous Use of Gesture

The use of spatial language in helping children develop difficult and abstract mathematical ideas is often enhanced by combining with the use of gesture (Congdon et al., 2017). The spontaneous use of gesture became explicit in many children's descriptions and provided the second insight into the way children engaged with the pilot tasks.

To illustrate how gesture was used and analysed across the range of tasks, Child 22's response to the Pilot Task 11: Hidden Fractions is revisited. This task asked children to consider, *'What fraction of the blue rectangle might be hidden under the orange square?'*

As previously presented in Figure 4.4 (reproduced below for convenience), Child 22 created a representation where they explained that up to 2-thirds of the blue rectangle could be hidden underneath the orange square. When asked to explain their written response (i.e., 2-thirds could be hidden), Child 22 flipped their hand from palm down to palm up (see Figure 4.19) while explaining their thinking, suggesting multiple mental transformations and iteration of the blue unit.

Figure 4.4

Child 22's Work Sample

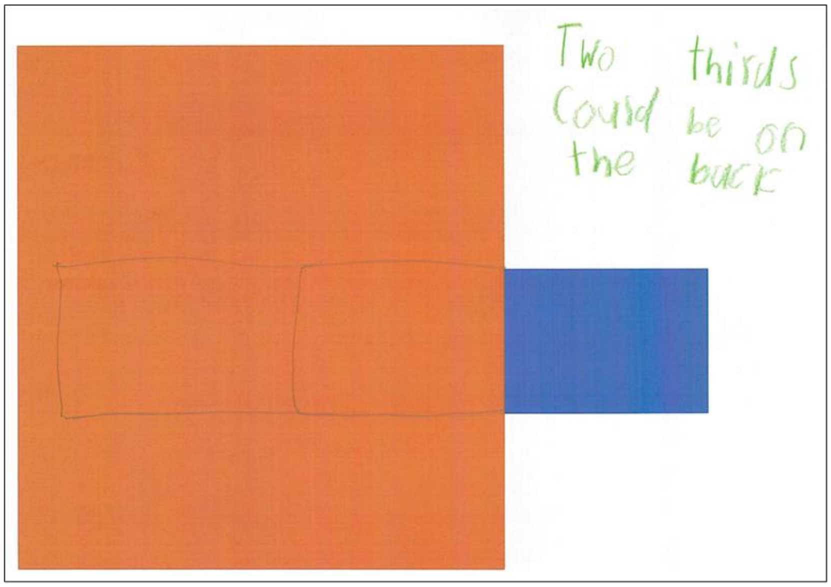
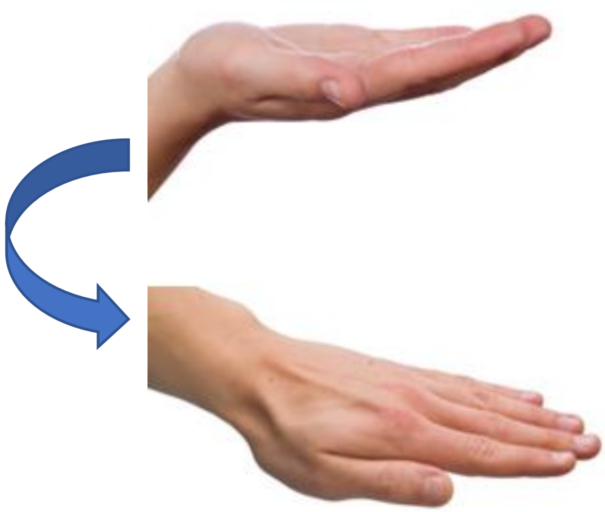


Figure 4.19

Example of the Hand Flipping Gesture Used by Child 22

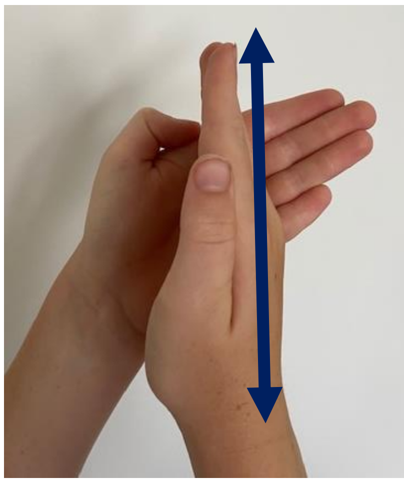


Child 22's gesture accompanied their description of 'flipping', suggesting they used spatial visualisation to iterate the unit fraction. This gesture, representing the iteration of a unit fractions, was also used extensively by children in Task 10: The dinosaurs have escaped.

Similarly, many children also used gesture to demonstrate and experiment with partitioning or 'cutting' the different routes or areas across the suit of tasks (such as cookies, roads, paths, carparks, etc.). In this gesture, the right hand would typically move in an action that replicated a sawing or cutting motion, indicated by the vertical arrow in Figure 4.20.

Figure 4.20

Recreation of a Cutting/Sawing Gesture Observed During Partitioning Contexts



This gesture was prominent in Pilot Task 8: Who ate more pizza?, which had several problems for children to work through, including:

- If you were really hungry and wanted the biggest slice of pizza, which pizza would you take a slice from: a pizza cut into quarters, or the same size pizza cut into eighths? Why?
- If you had to share a pizza between three friends, or two pizzas between four friends, which situation would give you more pizza? How do you know?

- Sam claimed he ate more than Victoria because he took two slices from the pizza partitioned into eighths. Victoria ate only one slice from the pizza that was cut into fourths. Who ate more? How do you know?
- What are some other ways you can cut your pizzas, so Sam and Victoria eat a *different number of slices* but eat the *same amount* of pizza?

Child 24 worked through the second problem, initially writing the statement '1 and 3' and '2 and 4' on their workbook. Child 24 then said, 'the first one is a third', and as they said this, drew the symbolic notation of $\frac{1}{3}$ in the air. Child 22 then said, 'but 2 pizzas, gives you… half bits', while drawing the symbol of $\frac{1}{2}$ in the air. Even with prompting to draw what they were thinking, perhaps as pizzas or with the symbols, the child could not communicate their thinking adequately. However, Child 22 appeared to be using the symbols of each unit fraction as an abstraction of the quantities involved, and, therefore, a metaphoric gesture that represented quantity.

Other examples of gesture were illustrated by Child 1 when describing their thinking for the fourth problem in Pilot Task 8. They explained that if a pizza is 'divided into half', that is one big piece each for Sam and Victoria. As Child 1 was explaining, they made a gesture like they were struggling to hold up a big slice of pizza to their mouth. The child then gestured picking up something between their two forefingers and thumb and moving their hand to their mouth like they were eating what they had gestured picking up in their hand. Child 1 repeated this picking up/eating gesture many times. Their explanation was that Victoria's half of the pizza was cut into 'a million little bits!', while using the sawing/cutting gesture very fast, when describing that Sam and Victoria still ate the same quantity of pizza.

Given these children's likely limited experiences with the range of fraction concepts and ideas presented in this series of tasks, the use of spontaneous gesture appeared to be an important vehicle for them to communicate their use of spatial reasoning in conjunction with their emerging fraction understandings. In relation to the data analysis for the teaching experiments, the code category of gesture was confirmed, however I made the decision not to implement gesture explicitly into the pedagogical approach of the intervention. The reason for doing so was twofold. Firstly, as reported in Chapter Two, there is very little literature concerning the way in which children 6 – 7 years of age utilised gesture in the development of fraction ideas and what literature does exist is not conducted in intervention-based, exploratory contexts. Secondly, there is also mixed evidence for what impact teacher led gestures have on children's development of an extended range of fraction ideas. As a result, I decided to explore the ways in which children spontaneously used gesture throughout this intervention, rather than being influenced by my use of gesture.

4.6 Chapter Summary

The pilot revealed that the suite of tasks was typically suitable and accessible for the target age group in this study. It was apparent that although many children had little understanding of or prior experiences with the three meaning of fractions, they could generally work with an extended range of fractions meanings. The children explored these ideas through a spatial reasoning approach, which provides further evidence that spatial reasoning has a positive influence on children's mathematical development in an area of mathematics that is not considered as typically 'spatial' in nature. The findings of this pilot suggest that a spatial reasoning approach is a viable pedagogy to continue to explore in the forthcoming teaching experiments.

The pilot also enabled the key indicators of the local instruction theory to be considered and revised. The modification of the first key indicator was in response to supporting a spatial awareness of the magnitude of each share, rather than children relying on more familiar processes such as counting. Therefore, the first key indicator was revised to include the *creation and justification of equal shares* in both discrete and continuous contexts, to promote a more general understanding about fair shares between the two contexts.

The second key indicator of *reinitialising the unit* appeared to be an appropriate and achievable learning goal for the children, which build on from the first key indicator of creating and justifying equal shares. No changes were made to this key indicator as a result of the pilot.

The third key indicator was initially hypothesised as the children developing the mental process of splitting through partitive division and recursive multiplication. However, the ability to visualise the act of splitting was foundational two the first two key indicators, so it was reconsidered. Furthermore, it was apparent that the children were able to discuss and justify early proportional equivalence in some of the associated mapping tasks. The third key indicator was therefore revised to *recognising proportional equivalence.*

While there was limited indications of children's ability to work with early multiplicative ideas such as simple ratio due to the timeframe of the pilot, there was sufficient evidence that they found the tasks associated with these ideas intelligible, and therefore the final key indicator of *connecting multiplicative relations* remained unchanged.

Finally, the pilot enabled me to identify children's use of spatial language and gesture as a means for identifying and interpreting children's engagement with various spatial reasoning constructs. Young et al. (2018) postulate that many complex and abstract mathematical topics—such as fractions—are inherently connected to the use of spatial language. This is in part because the generation of fractional parts (partitioning) and naming of such parts (unitising) can be described by the spatial transformations performed to generate such quantities, and the magnitude of such parts can be described using spatial dimensions and spatial prepositions. This is a critical insight for the present study because it reiterates how important the spatial context is to the development of early fraction ideas in each of the tasks.

Similarly, gesture, was another representation that provided insights into how children were engaging with spatial constructs and how they were connected to the fraction ideas they were exploring. Specifically, the cutting/sawing gesture indicated the act of partitioning that a child had imagined or was anticipating within the context provided because it was associated with descriptions of the quantities generated (and the size of the parts). Moreover, the flipping hand gesture indicated the children's mental manipulation of objects, quantities, and parts to generate new quantities, again suggesting spatial visualisation. Trafton et al. (2006) states that 'one of the fundamental findings within the gesture research community is that people gesture when they are thinking about something spatial' (p. 2). Therefore, identifying children's use of gesture and spatial language is important because of the spatial context that underpins the present intervention. While gesture was identified as an additional, valuable representation in which to interpret the children's thinking, I made the decision not to include gesture into my own pedagogical approach in the intervention. This decision was based on the limited literature that explores children's spontaneous use of gesture when learning fractions, and the mixed results that are reported from studies that explored the effect of teacher led gestures in instruction.

In DBR studies, the iterative analysis of the intervention is critical to understanding the phenomenon of the educational problem. The pilot phase of such studies is no exception. In the present study, this phase enabled insights into the performance of a sample of 6 – 7-year-old children on a range of tasks, thus enabling the design of the intervention program structure for the subsequent teaching experiments and the refinement of the local instruction theory. The teaching experiments are detailed in Chapters Five and Six.

Chapter 5: Teaching Experiment Insights – Class B

5.1 Chapter Overview

As described in Chapter Three, the teaching experiment phase of the study was designed to explore the proposed local instruction theory for developing an extended range of fraction meanings in the early years of school. The local instruction theory is presented in the form of a series of key indicators that describe the conjectured way fraction meanings and spatial reasoning are connected and are hypothesised to develop over time. As a manifestation of the local instruction theory, a suite of tasks was designed and trialled with small groups of children as described in the previous chapter. This enabled the key indicators to be refined and the full intervention program to be sequenced accordingly for the teaching experiment phase. This chapter presents the findings from Class B as the first iteration of the teaching experiment.

To provide the context for Class B, section 5.2 describes the classroom environment based on a series of pre-intervention classroom observations and an analysis of the children's pre-intervention Task Based Interview (TBI) results. Section 5.3 discusses the preparation of the teaching experiment in light of the TBI results. Section 5.4 analyses how children demonstrated each of the key indicators of the local instruction theory via the children's responses to the tasks in the intervention program. The purpose of this section is to examine the extent to which children can work with the three meanings of fractions—fraction as measure, fraction as an operator and fraction as a relation—when experienced through a spatial reasoning approach. To further explore the impact of the innovative approach, section 5.5 presents quantitative and qualitative analyses of the post-intervention TBI data. Section 5.6 summaries the insights from this iteration of the teaching experiment and their implications for the next iteration.

5.2 Setting the Scene: Class B

An important part of DBR is understanding the classroom context in which the innovative approach is being implemented. In the context of the present study, this is reflected by the constructivist and sociocultural perspectives as the orientating theories underpinning this study, whereby identifying children's existing knowledge and the context of the classroom are important factors when considering the impact of the intervention. To recap from Chapter Three, the Year 1–2 composite class participating in this first iteration of the teaching experiment was from a regional public primary school in South Australia. The class comprised of 23 children—16 boys and seven girls. The mean age was 6 years, 11 months. Data were collected in the form of pre- and post-intervention TBI for two reasons. The first was to determine the children's general understanding of whole number ideas expected for this age group, their understanding of the fraction meanings, and their spatial reasoning capabilities. The second was to determine if any significant shifts in children's whole number, fraction and spatial reasoning capabilities had occurred that could be attributed to the intervention. Children's work samples from each lesson throughout the intervention were collected to analyse how children represented their thinking in the various contexts. Observations and written reflections of each lesson from both the classroom teacher (Teacher B) and I (as the teacher/researcher) were collected to provide various viewpoints and interpretations of children's mathematical thinking.

5.2.1 Understanding the Classroom Environment

To become familiar with the classroom context, I observed Teacher B and the children in three mathematics lessons immediately prior to the intervention. The observations commenced in May 2019, which meant the children were in Term 2 of the school year. Teacher B stated that their planning was guided primarily by the content descriptors in the Australian Curriculum, but they also used educational websites such as Scootle (Education Services Australia, n.d.), a digital education newsletter the school subscribed to, and print-based materials they have collected that align with the Australian Curriculum content descriptors.

The first lesson I observed required the children to roll two, 0–9-sided dice, to represent each quantity with linking cubes or pop sticks, and then add these quantities through the use of a number of sentences and equations. The second lesson involved completing a series of worded problems, such as, *Jenny bought three cupcakes for $2 each at a cake stall. She paid the stall holder with a $10 note. How much change did she receive?* The problems were presented on the interactive whiteboard and the teacher read through each problem, modelling and suggesting strategies (such as drawing pictures, or writing equations like 2+2+2 = $6), before asking the children to complete the rest of the problem individually in their workbooks. Many children chose to draw pictures but were observed asking the teacher what to do or to re-read the problems throughout the lesson, as the tasks appeared challenging for many.

The third lesson I observed involved several worded problems written on the whiteboard, where the children were asked to model place value and trading procedures using base ten blocks. The Year 1 children worked with tens and ones, and the Year 2 children worked with hundreds, tens, and ones. Teacher B stated that the intention of this lesson was for children to use the base ten materials to model the problem and then record it in their books using a vertical algorithm. Many children were observed playing with the materials (stacking, lining up the blocks) rather than completing the problems. Few children attempted the vertical algorithm in this time as it appeared too difficult.

As described in Chapter Three, the CLASS (Pinta & Hamre, 2009) framework was used to analyse and understand the typical ecology of the classroom. Using this framework, the following themes were dominant. There was a strong emphasis on the *instructional support* *domain*, specifically for developing strategies to support *content knowledge* in the form of modelling and repeating the language that was associated with procedures. For example, children were encouraged to describe and practice the abovementioned cupcake problem with several variations (i.e., starting with one cupcake and working out the change from $10), using equations when adding and representing their responses. The *quality of feedback* dimension was evident to some extent in the way the teacher expanded on different children's responses (and performances) to promote an understanding of the procedure. Teacher B would often direct a child to 'look how … has solved the problem', encouraging the children to share and model their strategies. Over the course of the three observations, it was clear that the children understood the expectation that they were to record their solutions to all problems, even if they were primarily working with concrete materials. However, many children did not complete this requirement.

There was less *open-endedness* and *creativity* observed with regard to the *concept development* dimension, as children were scaffolded more explicitly to complete the problems in a consistent way. For example, many children were observed asking Teacher B throughout the third lesson, 'what do I write?', after adding the quantities with base ten blocks because they seemed unsure of the problem and how the materials used were to be translated into a vertical algorithm.

These observations indicate that the children were not confident in making connections between concrete, pictorial and symbolic representations. I asked Teacher B about their pedagogical approach, and they suggested that the children 'needed lots of repetition to really understand the problems and getting their sums recorded'. As there was little emphasis on formal symbolic notation within the tasks designed for the present intervention study, I anticipated that the children would require time and explicit scaffolding to develop and explore the connections between modes of representations that may be less familiar to them.

5.2.2 Pre-Intervention Task-Based Interview Insights

The pre-intervention TBI was conducted one on one and comprised 24 items, divided into three sets. The first nine items were whole number based, to assess aspects of two big ideas in numbers: Trusting the Count (Set One) and Place Value (Set Two), created by Siemon (2006). The remaining 15 items (Set Three) targeted children's fraction understanding and spatial reasoning capabilities. I created or adapted these items based on the literature concerning both fraction and spatial reasoning research (see Chapter Three). Many of the tasks were open-ended, which meant children's partially correct responses were captured (in addition to correct and incorrect responses). The raw scores for the TBI are in Appendix H.

5.2.2.1 Set One: Trusting the Count Insights

The items for Set One are summarised in Table 5.1.

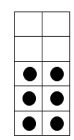
Table 5.1

Set One Items

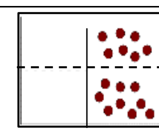
Item 1: Subitising cards
Cards 1–6 (common dot die arrangement)
Cards 7–10 (tens frames and structurally ordered arrangements, e.g., triangular arrangement of dots for 10)
Cards 7–19 (tens frames ordered and random)

Item 2: Hidden counters task
Place five counters and bag in front of child, rattle to demonstrate that there are counters in the bag. Place four counters in front of child.
'There are five counters here and five more in this bag. How many counters altogether? How did you work that out?'

Item 3: Tens frame bananas
Children are asked to think about the dots on the tens frames as bananas.
'If I have this many bananas, and three more bananas were added, how many are there altogether?'



Item 4: Hidden dots task
'There are seven dots here (in the top section) and nine dots here (bottom section).'
Cover the '9' card with the flap and ask, *'How many dots altogether? How did you work that out?'*



In Item 1, 21 children demonstrated the ability to subitise consistently to at least 12 with ordered and unordered dot arrangements, with three children identifying quantities from one to 19. Similarly, 16 children answered Item 3, with 11 demonstrating flexible whole number knowledge of four less than 10 is six, adding three makes nine. What was evident in the majority of children's responses to these items was the use of a residual thinking strategy (Cramer et al., 2002). Residual thinking is described as recognising the quantity or amount missing to complete a whole (Clarke & Roche, 2009). As the subitising cards with the collections of 14, 17, and 19 were presented in tens frames, several children stated they subitised the smaller collection of empty squares (or amount required to make to the next ten) and subtracted from the whole. That is, when 17 was presented for children to subitise, the children recognised three empty squares within two tens frames, meaning they recognised three less than 20 is 17. Similarly, in Item 3, the majority of children described ‘seeing’ three more than six is the same as one less than 10.

Items 2 (Hidden counters) and 4 (Hidden dots) proved more challenging; 10 children counted on instead of recognising five and four is double five, less one; and only seven children fluently stated five plus four is nine, with little regard for the physical objects. Similarly, in Item 4 (Hidden dots), the majority of the eight children who were able to engage with this task predominately counted on by ones using fingers or making marks on a page, rather than demonstrating fluent part-part-whole knowledge for nine and seven. These observations suggested that the children are figural counters (Steffe et al., 1983), where the children required the use of visual cues (such as fingers to count or tapping to keep track) or audio cues (counting aloud) to keep track of hidden collections.

An insight revealed from Set One is that despite the vast majority of the children demonstrating part-part-whole knowledge and conceptual subitising capabilities in Items 1 and 3, they had limited success in items where collections were hidden or randomly arranged. This suggests that their success may be connected to the physical structure and arrangement of the collections; that is, when children were required to engage with collections where there were no structural aspects to the representations—such as Items 2 and 4—the children did not perform as well as in Items 1 and 3. This was surprising, particularly for Item 2 (Hidden collections), as many children had demonstrated fluent part-part-whole knowledge in Item 3 (Ten’s frame bananas); yet, when the was no structure or the parts were hidden, the children seemed less confident and competent.

5.2.2.2 Set Two: Place Value Insights

The items for Set Two are summarised in Table 5.2.

Table 5.2

Set Two Items

Item 5: Counting 26 counters Child counts collection and records. Circle the ‘6’ in ‘26’ and ask, *‘Does this have anything to do with how many counters you have there?’* Circle the ‘2’ in ‘26’ and repeat the item. Ask the child to explain their thinking if not obvious.
Item 6: Place-Value Bundles 13 bundles of ten pop sticks and 16 single sticks are provided. Child is asked to make 34 using these materials.
Item 7: More than/Less than… Card with ‘86’ is presented to the child. *‘Write the number that is one more than this number. Write the number that is one ten more than this number.’* If correct, say, *‘Write the number that is three less than this number. Write the number that is two tens more than this number.’* Ask child to explain their thinking if not obvious.
Item 8: Proportional number line task Place the 0 to 20 open number line card in front of the child and say, *‘Use the pencil to make a mark to show where you think the number eight would be. Why did you put it there?’* Repeat with the number 16. If reasonably accurate and/or explanation plausible, turn the card over to show the 0 to 100 open number line and say, *‘Make a mark to show where you think 48 would be. Why did you put it there?’* Repeat with the numbers 67 and 26. Ask child to explain their thinking if not obvious.

In Set Two, five children demonstrated an understanding of place value units by their ability to name, compare and rename collections in terms of their parts evidenced in Items 5 and 6. For example, two children described 26 as two tens and six ones in Item 5 (26 Counters), and three children responded to Item 6 (Place Value bundles) by describing 34 as three tens and four ones. When the children used the counters in Item 5 (26 counters), they took more time to name the '2' in 26 as two tens. This suggested that this small number of children saw 10 as a composite unit, in that they could either describe (or were observed) counting each bundle as '1', where one was considered a unit of 10 (i.e., one ten, two tens, etc.). Cobb (1995) categorises this thinking as an abstract composite unit. This is the ability to see the units of 'one ten' and 'ten ones' concurrently which is an early indication of multiplicative understandings about our number system (Rogers, 2012).

The majority of Class B (16 children) demonstrated they were able to count a collection of 26 objects accurately but were unable to recognise the value of the digits in this number beyond stating '20' and/or 'six'. Similarly, 11 children were able to represent 34 as three bundles of ten and four ones, but they too demonstrated an additive understanding of the two quantities (i.e., 30 and four is 34; Thomas, 2004). Eight children initially wanted to count only the ones provided to make 34, unbundle the tens to check the count of the sticks individually, but then counted the bundle as ten rather than considering them as 'one ten'. This behaviour suggested that these children considered the unit of 10 as a numerical composite unit (Cobb, 1995) rather than an abstract composite unit. As described by Cobb, this means the children count in tens by using a count of ones (10, 20, 30, etc.), only seeing the unit as '10 ones', as opposed to simultaneously seeing the unit of 'one ten' and '10 ones'.

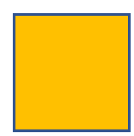
This was an important insight for this intervention and how children may perceive fractions, as the ideas of composite and countable units require children to think flexibly about partitioning, creating unit fractions and renaming fractions (Kieren, 1995). For example, just as 12 could be made of six and six, eight and four, 10 and two, and so on, fractions can be represented as the sum of other fractional amounts, such as one can be the sum of 1-half and 2-quarters (Kieren, 1995). This analysis of how the majority of the children viewed units in whole number contexts may indicate how they may establish an understanding of quantity and magnitude in fractional contexts.

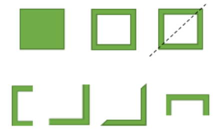
5.2.2.3 Set Three: Fractions and Spatial Reasoning Insights

In Set Three of the pre-intervention TBI, there were only four items where half or more of Class B scored a partial or correct response: 10 (Folding fractions), 16 (Halving the stars), 18 (Gisele's paper square) and 19 (Scale the picture). All of these items focused on doubling and halving, in addition to spatial visualisation or spatial proportional reasoning. The full list of items in Set Three is in Appendix B. The four items discussed here are summarised in Table 5.3.

Table 5.3

Set Three Items

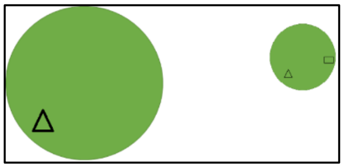
Item 10: Folding fractions	
Child is shown the image of a square. *'How many ways can you imagine folding a square in half? Can you describe what you think it would look like if you folded it in half, then in half again? What is each part called?'*	
Item 16: Halving the stars	
Child is presented with the image. *'If you gave away half of this collection of stars (16), how many would you have left?'*	
Item 18: Giselle's paper square	



A series of folds is made to a square, and the child needs to identify what the end result would be from four possible options. The child is only shown the four possible outcomes to choose from.

'Gisele had a green sheet of paper and cut a white shape out of the middle of the paper. Then she folds the paper in half, diagonally. Which of the four shapes below did Gisele see?'

Item 19: Scale the picture



An image of two circles is presented. The diameter of the smaller circle is half that of the larger circle. The smaller circle contains two shapes, a triangle and a rectangle. The larger circle only includes the triangle drawn to scale. The child is required to draw the missing rectangle in the larger circle and describe the difference in size between the two shapes.

'Can you complete the picture of the circle on the left so that it has the same shapes as the circle on the right? Explain why you chose to draw the shapes in that way.'

For Item 10, 13 children identified horizontal, vertical, and diagonal folds as multiple ways of folding the paper in half. These 13 children could also determine four parts would be created in the second part of the question; however, they did not name the parts as quarters and fourths but did indicate the parts were equal in size. For example, Child 38 said, 'it makes a set of smaller squares all the same', and Child 44 'there's four squares the same, in the big square [of paper]'. Only one child accurately named all of the fractional parts created, stating that 'as long as the four parts are the same size, they're quarters', referring to creating quarters by repeated halving diagonally and vertically, suggesting an understanding of partitioning. What was evident about the children's general engagement with this task, was an awareness of the structure, arrangement and congruent nature of the parts generated from repeated halving. In addition, the spatial contexts of the task (which are likely to be unfamiliar to the children, given the expectations of the Australian Curriculum at this level) did not seem to add another layer of complexity to the tasks. That is, the majority of the children were able and willing to engage with visualising the outcome of folding the paper, even if they could not complete the fraction component of the problem.

Item 16 revealed the children’s problem-solving strategies were typically influenced by their awareness of the structure and arrangement of the stars. Seven children confidently and without hesitation stated, ‘eight is half’, saying they could ‘see double eight’ in the image or pointing to two columns of four stars to justify their answer. Seven other children immediately recognised the column structures but individually counted one or two columns to determine eight was 1-half. This suggested they were paying attention to the geometric symmetries and structure of the collection but not viewing the relationship between the 16 starts and 1-half of 16 as a fraction as an operator problem; rather, they were using a counting approach to solve the item. Given the intervention incorporates whole and rational number ideas simultaneously in the intervention (i.e., specifically when partitioning discrete collections), it will be important to ensure there are explicit and intentional teaching opportunities for children to connect these ideas from a partitioning basis, rather than a counting approach.

Of the 14 children who correctly identified the outcome of cutting and folding a piece of paper in Item 18 (Gisele’s paper square), 12 children made explicit comments about the half being a triangle as a result of folding the paper, or that two triangles make the (original) square. These children demonstrated they can engage with spatial visualisation to determine the correct outcome, but they also paid explicit attention to the geometric symmetry and structure of the part. This awareness of symmetry and geometric structure was prevalent in children’s engagement with this set of items and may assist children to develop their further understanding of fraction ideas throughout the intervention.

This item also revealed the highest instances of gesture across the whole TBI; 11 children put out their hand palm side up and flipped to palm side down (or similar) when explaining how the part they chose was generated. Eight children also used their hand to signify a diagonal fold across the paper to justify the geometric structure and regularities of the shape in the way they visualised the process and outcome.

Item 19 (Scale the picture) required the children to use spatial proportional reasoning to draw a missing rectangle to scale using an image as a referent. Only two children described the relationship between the two images with conceivable explanations in addition to correctly drawing the missing rectangle to scale. Child 46 stated 'it's the same shape, just zoomed in twice as big [pointing to the lager circle]'. Child 42 stated, 'the missing square is actually the same thing [referring to the rectangle on the smaller circle], this is just twice as big, so you have to make it all look right'. These statements suggested these children paid explicit attention to preserving the proportional relationship between each image, indicating an awareness of proportional equivalence. Seventeen children were able to accurately draw the scaled shape but were unable to articulate a relationship between the two images, indicating a partial understanding. Drawing the image accurately suggested they have some appreciation of proportion in this problem but do not necessarily know how to describe the relationship. This had important implications for the intervention in terms of examining children's work samples where their verbal, gestural, or written descriptions may be absent or not captured in each individual task. Thus, the one-on-one TBI was reconfirmed as an important tool for holistically comparing each child's performance against the work samples generated in the intervention.

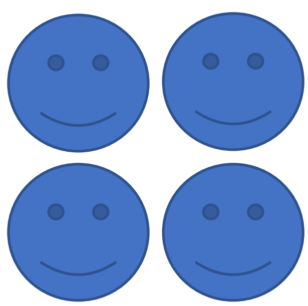
5.2.2.4 Summary of Pre-Intervention Task-Based Interview Insights

Across the pre-intervention TBIs, it was clear that the children have not had the opportunity to experience many of the ideas and contexts in the TBI, resulting in overall limited success across the TBI (see Appendix H). However, this data collection method revealed a new insight for consideration in the intervention and local instruction theory more broadly. In the analysis of children's responses to the Set One subitising tasks and the residual thinking analysis, the children seemed to be intuitively drawn to the structural regularities of different representations—either those presented as stimulus or in their own pictorial representations and gestures. For example, a number of children identified a relationship between quarters generated from repeated halving through seeing a structure between the parts and the whole in Item 10. The structure was referring to the geometric congruence of the quarters created within the larger page. Further, this structure was observed when children referred to the column arrangement of the stars in Item 16 to determine half.

The structural element in the children's descriptions and representations was evident in other items. For example, in Item 15 (Missing faces), the four circular faces representing 2-thirds of the whole set was presented in a two-by-two array (see Figure 5.1).

Figure 5.1

Item 15: Missing Faces Representation



The three children in Class B who correctly identified a unit of two additional faces as the missing fractional part either (a) drew a ring around two faces to explain how they 'saw' the structure of each third (i.e., two faces as a unit), or (b) in conjunction with gesture (where they used their hand in a cutting/ sawing-like action), partitioned the group into thirds using a row-and-column-like formation to signify where the missing third was to go. While only a small percentage of the class succeeded in this task, the fraction problem is quite complex for this age group (which has typically have not yet explored thirds). Attention to the pattern and structure of the representation appeared to be an important construct for understanding this problem.

When analysing the data from this pre-intervention TBI with a spatial structuring lens, it was apparent this spatial skill was also successfully used by two children in Item 14 (Bags of wool). Although 20 children responded incorrectly, the children who were successful demonstrated clear use of spatial structure in their representations. For example, the children who demonstrated an understanding of the problem provided representations that depicted clear structural elements to represent the referent unit (see Figures 5.2 and 5.3).

Figure 5.2

Child 35's Work Sample

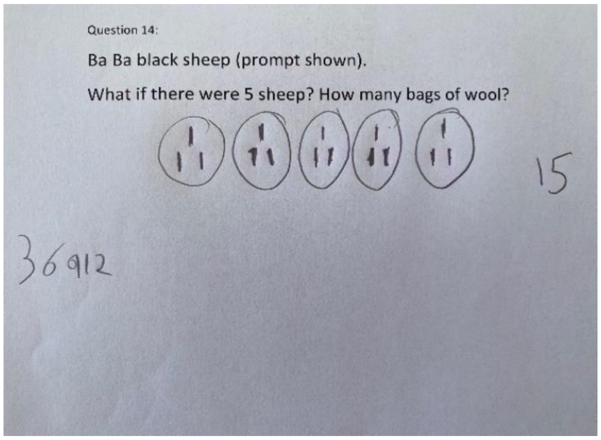
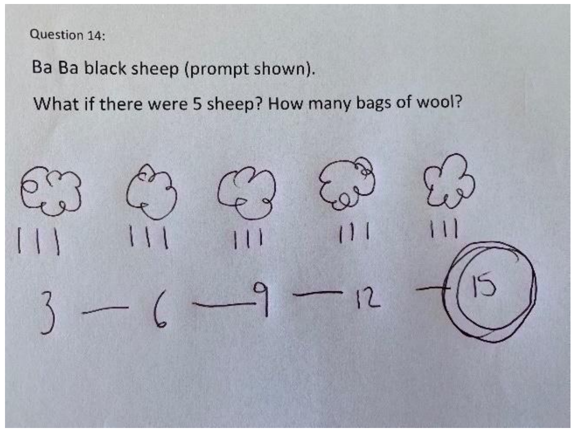


Figure 5.3

Child 46's Work Sample



Children 35 and 46 organised the tally marks, indicating a use of spatial structure, suggesting they paid deliberate attention to the structure, pattern, and arrangement of the objects. Child 35 presented a row-and-column-like arrangement, in addition to a number line–like figure used to represent the numerical pattern. The use of structure indicates two interpretations of fractions: the sheep and tally marks indicate a many-to-one idea (fraction as relation) representation, and the number line indicates a many-as-one (fraction as measure) idea. Child 46 drew a triangular formation of each group of three bags of wool to represent the relationship between the number of sheep and the three bags of wool produced (a many-as-one representation). The awareness and overt attention these children paid to the structural elements of the representations throughout the TBI appeared to be an important aspect of their success. This finding is discussed as an implication for the teaching experiment in the next section.

5.3 Implications for the Teaching Experiment

Analysis of the classroom observations and children's pre-assessment TBI responses provides two implications for this iteration of the teaching experiment: (1) the consideration of spatial structuring as an additional spatial reasoning focus for the intervention and (2) how this additional construct may assist in refining the local instruction theory and intervention program more generally.

5.3.1 Spatial Structuring Considerations

The interpretation of spatial structuring in children's pre-intervention TBI engagement provides additional insight into children's responses throughout the intervention program. Mulligan and Mitchelmore (2009) describe spatial structure as an awareness of mathematical relationships that are supported by spatial patterns and arrangements. For example, spatial structuring is a core element of subitising because it draws on the patterns and arrangement of objects to perceive the quantity of collections. This is evident when objects are arranged to represent geometric shapes or arranged in familiar patterns—such as those on a traditional dot die, or patterns presented in five and ten frames (Mulligan & Mitchelmore, 2009; Young-Loveridge, 2002). With regard to the present study, the children's responses and representations suggested that spatial structuring could be a key construct in how children develop their understanding about the fair share idea (fraction as operator), distribution and many-as-one ideas (fraction as a relation) throughout the local instruction theory, particularly in discrete sets, given the connection to subitising. It also suggests that it is a critical construct that promoted a partitioning approach, rather than a counting approach, because of the focus on viewing discrete collections as a set of composite units.

Spatial structuring has been studied extensively by Mulligan and Mitchelmore (2009), in which children's responses to a wide variety of studies investigating children's patterning, counting, the numeration system and multiplicative thinking were assessed. Their research resulted in the development of the Pattern and Structural Assessment (PASA) classification framework (see Table 5.4), which is used to characterise the level of spatial structure evident in children's representations of specific tasks.

Table 5.4

Pattern and Structural Assessment (PASA) Classification Framework (Mulligan et al., 2020)

Response category	**Characteristics of responses**
Advanced structural	An accurate, efficient, and generalised use of the underlying structure.
Structural	A correct but limited use of the underlying structure.
Partial structural	Shows most of the relevant features of the pattern and structure but inaccurately organised.
Emergent	Shows some relevant features of the pattern and structure but incorrectly organised.
Pre-structural	Shows, at most, limited and disconnected features of the pattern and structure.

The intention of this framework was to analyse children's awareness of pattern and structure in mathematical tasks explored extensively in the associated Pattern and Structure Mathematics Awareness Program (PASMAP; Mulligan & Mitchelmore, 2016) in relation to patterning and early algebraic thinking. However, this framework can serve as a practical and consistent tool to analyse children's awareness of pattern and structure in the present study to determine if and to what extent the use of spatial structure is related to the children's understanding of rational number ideas.

Patterning and multiplicative thinking share underlying concepts of partitioning, unitising, and equivalence; therefore, this framework lends itself to adaptation to help analyse the use of spatial structure within fraction contexts (Confrey et al., 2014b; Nunes & Bryant, 1996; Papic et al., 2011). The warrant for adapting this framework is to determine if and how children's varying awareness of spatial structure may influence or is connected to specific understandings of fractions. Table 5.5 describes how this framework was adapted for the present study, with examples specific to the context of whole and rational number ideas.

Table 5.5

Spatial Structuring Codes Adapted for This Study

Code category	Code(s)	Description	Examples or provocation
Spatial structuring	*Advanced structural*	Generalised use of the underlying structure.	The use of array-like structures to generalise part-whole/part-part relationships. Identification of proto-ratio representation of core unit structure. Transfers fraction meaning between continuous and discrete contexts.
	Structural	Correct but limited use of the underlying structure.	Identifies an equal grouping and or distribution within fraction context. Represent a relationship between number of parts and size of part.
	Partial structural	Shows most of the relevant features of the pattern and structure but inaccurately organised.	Demonstrates attention to equal grouping/distribution for set collections, in unit size of continuous fractions but is inconsistently represented/described.
	Emergent	Shows some relevant features of the pattern and structure but incorrectly organised.	Some attempt at equal distribution/ grouping/unit size but quite unorganised or misrepresented.
	Pre-structural	Shows, at most, limited and disconnected features of the pattern and structure.	Unorganised and unequal understanding of unit size/form and distribution for a fair sharing context. Represents non-mathematical/irrelevant features of the problem.

Based on this framework adapted from Mulligan & Mitchelmore (2009) the code category of spatial structuring and their associated codes were added to the codebook for thematic analysis. The implications of this finding will now be discussed in relation to the local instruction theory.

5.3.2 Refining the Local Instruction Theory

While the children displayed a minimal understanding of the three meanings of fractions in TBIs, they did demonstrate the level of understanding about whole number and place value ideas expected for this age. The children were able to communicate and represent their thinking in various ways, including engaging in spatial reasoning tasks that were less familiar. Given the emphasis in this intervention program was on developing the relationship between fraction meanings and spatial reasoning (not symbolic notation), it appeared the tasks would be accessible to the children. A small number of children did demonstrate some accurate, early understandings of the various fraction meanings through use of spatial reasoning strategies; this suggests the current local instruction theory and subsequent intervention program developed in Chapter Four is suitable and achievable for this age group. However, the examination of spatial structuring as an additional spatial reasoning construct suggests it is a valuable inclusion into the local instruction theory and subsequent teaching program. As a result, the spatial reasoning description for the key indicators of the local instruction theory have been refined to include this construct (see Table 5.6).

Table 5.6

The Revised Local Instruction Theory (Version Three)

Key Indicators	Characteristics of Tasks	
	Primary Fraction Foci	**Spatial Reasoning Foci**
Creating and justifying equal shares	**Fraction as Operator:** Fair shares Doubling/ Halving Partitive division/ recursive multiplication, Geometric Symmetries, Similarity **Fraction as Measure:**	Visual perception of equal groups (drawing on spatial structures and arrangements). Equality of parts regardless of model (i.e., equal parts for discrete collections and continuous models less than and greater than 1). Visual awareness of the geometric properties of parts and sets (e.g., shape, orientation,

	Many-as-one, Measure, Composite units, Unit fraction	pattern, symmetry). Observing the physical transformations of partitioning (dividing and reassembling), and visualising and predicting the outcome of a nominated split (e.g., spatial visualisation)
Reinitialising the unit	**Fraction as Measure:** Composite units, Unit fractions, Part-Whole fractions, Equivalent fractions **Fraction as Operator:** Fair shares, Doubling /halving; Partitive division/ recursive multiplication, Times-as-many, Similarity **Fraction as a relation** Many-to-one Distribution	Visualising measures between parts and wholes, and between composite and unit fractions through unitising. Exploring the spatial structure and arrangement of objects and sets to create and compare different unit fractions. Visualising magnitude relations between parts (double/ half/ times as many) the distribution of parts to determine equivalence.
Recognising proportional equivalence	**Fraction as a relation** Distribution, Proto-ratio, Equipartitioning multiple wholes, **Fraction as Operator** Doubling /halving, Times-as-many, 1-nth-of...…, Scaling, Geometric symmetries, Similarity **Fraction as a measure** Composite units Unit fractions Equivalent fractions	Visualising the relationship between equivalent measures, of same and different wholes.
Connecting multiplicative relations	**Fraction as Relation** Many-to-one, Distribution, Proto-ratio	Early relational understandings between the structure of part-part and part-whole quantities. (e.g.,

	Fraction as Operator Doubling / halving, Partitive division/ recursive multiplication Times-as-many, 1-nth-of…, Scaling	visualising and justifying the relational magnitude of fractions in relation to other fractions (e.g., a quarter is a half of a half/ twice as small); and working flexibly with non-symbolic simple ratios (e.g., 1:2 = 2:4).
	Fraction as Measure Composite units Part-whole fractions Equivalent fractions	

As part of the iterative analysis and refinement synonymous with DRB, spatial structuring was included as a focus in the tasks of the intervention program, described in Appendix C.

5.4 The Teaching Experiment: Insights from the Intervention Program

This section presents an analysis of how the intervention program provided opportunities for children to develop the three meanings of fractions. The analysis highlights the ways in which spatial reasoning strategies, and different forms of representations, were critical to the children's success in the development of fraction and ratio ideas in this intervention program. Children's responses to various tasks throughout the intervention, are discussed in relation to the key indicators of the local instruction theory.

5.4.1 Creating and Justifying Equal Shares

As detailed in Chapter Four, the first four lessons of the intervention explicitly focused on developing the key indicator of creating and justifying equal shares in both discrete and continuous models. The fraction as an operator meaning was foundational to this key indicator and promoted through the ideas of creating fair shares, identifying doubling and halving relations between the size of measures and number of parts created. In addition, partitive division/recursive multiplication was explored to help connect the idea that many parts can be generated and, therefore, named as a measure. The fraction as a measure idea of many-as-one was of focus to develop children's understanding of creating and justifying equal shares. Across the four lessons related to this key indicator the composite unit, and unit fraction ideas from the fraction as a measure meaning were also of focus in both discrete and continuous models. These ideas were tightly connected to the geometric symmetries and scaling ideas (fraction as operator) when exploring the outcome of creating different fair shares.

The primary spatial reasoning focus of this indicator was to promote a visual awareness of quantity; that is, recognise and compare visually the equality and size of the parts created, including visualising the size of a part in relation to the whole, and the outcome of creating different fair shares from the same whole. This drew on an awareness of the spatial structure assigned to the representations and concrete models such as geometric symmetries and similarities, and array-like structures for discrete collections. Examples of how children established this key indicator will now be discussed through a selection of activities from Lessons 1 to 4.

5.4.1.1 Lesson 1: Sharing Cookies

The children were asked to collect 12 plastic counters that represented 'cookies' to explore how many ways they could share them fairly, and how they knew each context represented a fair share. A worksheet was provided to assist the children in the fair sharing component of the lesson (see Figure 5.4).

Figure 5.4

Worksheet Provided for the Fair Sharing Component of Lesson 1: Sharing Cookies

The Doorbell Rang by Pat Hutchins

Share the 12 cookies equally between the number of children arriving at Ma's house.

12 cookies, 2 children	12 cookies, 4 children
12 cookies, 6 children	12 cookies, 8 children

An interaction I had with a small group of children who were unsure where to start with this lesson, highlights the way this lesson promoted this key indicator. Using six counters placed in two rows of three on the floor in front of the children (sitting in a circle) I asked the children, 'How many cookies are here?' and, importantly, 'How do you know?' At first, it was evident that the children counted the individual counters, possibly because this is what the children thought they had to do when asked such a question. After observing the children nodding and pointing at the counters to signify an individual count, they all said, 'six'. I asked, 'Is there another way we could have worked out there were six counters there, without counting each one?'

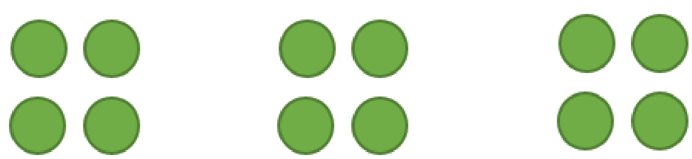
Child 27, who was pointing in the air to represent the individual counters to enumerate, immediately replied, 'It's lined up as three and three which means six'. Child 39 then stated, 'You can also tell it's six because you can see two, two and two, which is six'.

This suggested these children were aware of these part-part-whole relationships, perhaps even that they have a sound structural understanding of 'six' because of the way they were able to define its part-part relationships of six as 2-threes and 3-twos. I pointed out to the children that although no counters had been added or removed from the set, this structure provided two ways six counters could be visualised as equal shares. The children were prompted to think about using this knowledge when sharing the 12 counters, and set to work independently.

The children in this group started to make three groups, distributing one counter to each pile at a time—indicating the partitive division idea from the fraction as an operator meaning. They discussed how many counters there were in each group. For example, 'There are four, four and four because, I can see a square in a group of four counters, my counters make three squares!' (Child 45). Child 45's arrangement is represented in Figure 5.5.

Figure 5.5

Child 45's Representation of Sharing 12 Counters Between Three People



This representation and accompanying description indicate a many-as-one (fraction as measure) idea, and the partitive division/recursive multiplication (fraction as operator) ideas since they could see the structure of the units within the whole set. The child referred to the equality of the shares being based on the geometric property of the units, rather than counting to check they were numerically equal, demonstrating an early appreciation for partitioning.

Figures 5.6 and 5.7 are representations typical of what the children produced independently after this short, small group scaffold.

Figure 5.6

Child 33 Work sample

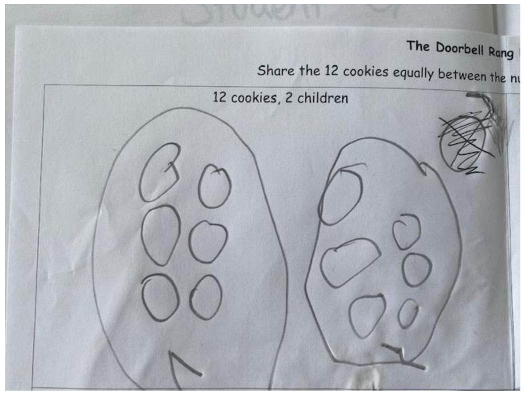
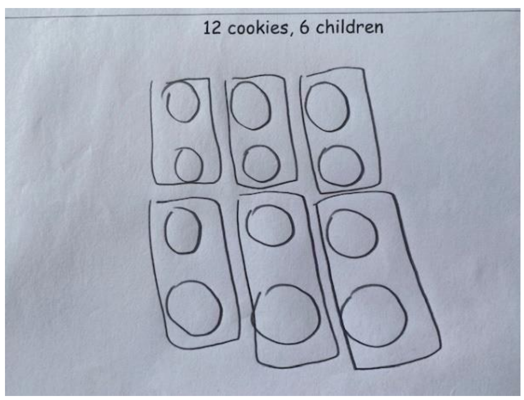


Figure 5.7

Child 30 Work sample



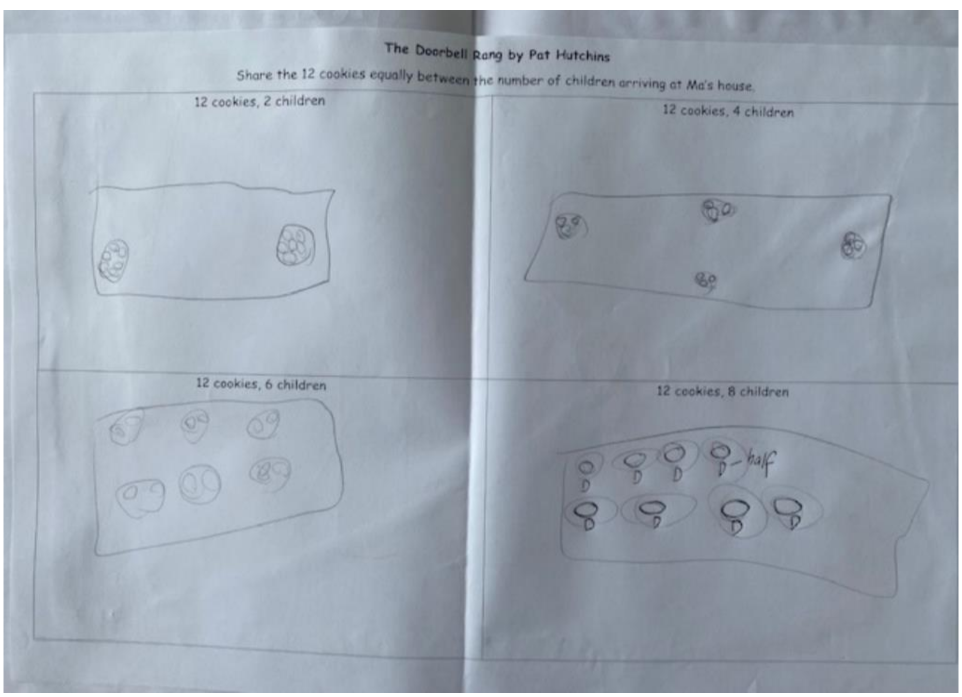
Children 33 and 30 demonstrated an awareness of structure and pattern in the arrangements they created in their representations. Child 33 stated they 'just knew' that two groups of six cookies is how to share 12 fairly, even though they were observed dealing one counter one at a time into two groups to exhaust the collection, suggesting partitive division (fraction as operator) understanding. This suggested Child 33 was aware of some structural features of the relationship between the equality of the parts created and the whole collection, though unable to provide a description of their representation.

Child 30 said they knew that if the problem involved six children and six cookies to share each person would get one, so they just doubled each person's share to two cookies each, because 'there was double six cookies to share'. Accompanying Child 30's description was the use of gesture, in the form of the child placing one of their hands on top of the other when describing the double of six. This suggested Child 30 understood the relationship between the quantities through the fluent description of doubling and halving, and how this was reflected within their representations. This is evidence of early fraction as an operator meaning and an appreciation of simple ratio, which is a fraction as a relation meaning.

In the next example, Child 39 chose to represent each problem with a rectangle as the kitchen table, larger circles placed on this 'table' indicating dinner plates and smaller circles drawn on the 'plates' representing the cookies and share each person would receive (see Figure 5.8).

Figure 5.8

Child 39's Representation

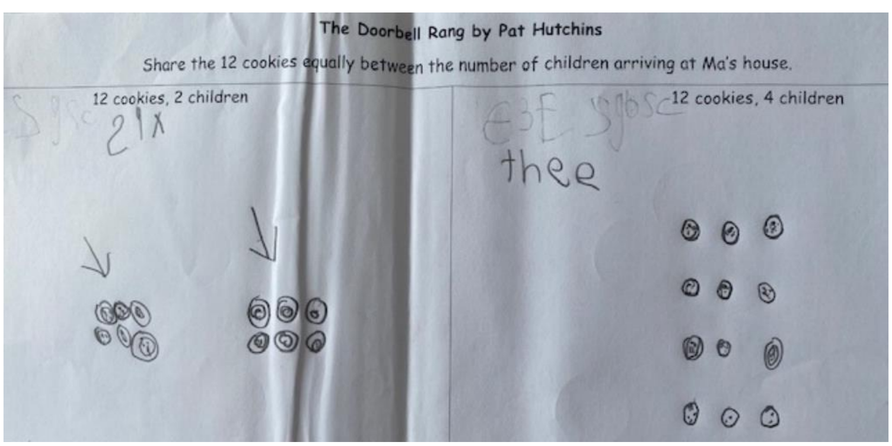


There is a consistent structure in the way the cookies are organised on the plates, and the position of the plates themselves, indicate that attention was paid to representing an order and regularity for how they are placed on the table in the creation of equal shares. When asked what 1-quarter of 12 cookies was, Child 39 described that there had to be four plates (on the table, referring to the rectangle in Figure 5.8). Each plate had three cookies; however, the child had to count and check that there were 12 in total, rather than recognising the multiplicative relationship between four threes (as composite units) represented. This response was considered to demonstrate a structural awareness because there is an appreciation that different amounts of cookies create a unit (many-as-one idea).

Similarly, Child 27's representation (see Figure 5.9) gave further insight into the way the structure of fraction units was used in this task. In the first problem (12 cookies between two children), Child 27 drew the cookies (without the use of concrete materials) in an array by distributing one of the 12 cookies at a time, indicating a partitive division approach.

Figure 5.9

Child 27's Work sample



In the second problem (12 cookies between four children), Child 27 chose to use counters first, and dealt one counter out at a time to four groups on the mat in front of them. However, the distribution of the counters was quite random, and the child repeatedly recounted each group they had made to 'check' they were all the same. When asked to represent this in their books, they said it was 'easier' for them to 'see' the groups (presumably meaning the equality and regularity of the parts) when they drew them in a four-by-three array or, in their words, 'in lines'. However, this too took a long time to represent, as Child 27 initially drew four cookies vertically, stating that was one group and that they had to then draw four more groups 'the same' (which would have resulted in a set of 16 counters). I asked Child 27 to check their logic with the counters, and with the assistance of a peer, they were able to conceptualise the groups (of four) would receive three cookies each. It is evident that Child 27's understanding is an example of structural representation for this context; there is a demonstration of fair sharing through partitive division ideas, because the child started with the intent to share in groups of four, but had some difficulties representing this quantity accurately. Importantly, Child 27 referred to the regularity and structure within the representation as the tool that enabled them to eventually create the fair shares in this activity.

5.4.1.2 Lesson 4: Sharing Divisible Collections

During Lesson 4, children were asked to provide some discussion about the ideas they had engaged with so far in this intervention. The explicit focus on the development of partitioning (specifically, fair sharing) was foundational to the first four lessons, where children explored partitioning of both continuous wholes (such as strips of paper, string, and paper circles) and sets of discrete objects (counters or cubes). In the whole class Launch phase of Lesson 4: Sharing easily divisible collections, the children were asked to describe what they learned over the recent lessons, specifically, '*What do you notice about the number of shares you create, and the size of those shares?*'

The responses included descriptions that used the following terms and phases (captured in the journal notes of Teacher B and myself):

- ‘Half can be in groups, or you can split something (whole) into half’.
- ‘Groups and wholes can make fractions like half’.
- ‘Groups are equal to be a fair share’.
- ‘A fair share means the same size of parts/groups/things’.
- ‘You get less when there are more to share between’.
- ‘More shares mean a smaller group (share)/more people mean smaller shares’.

During these explanations, nine children were observed using gesture to support their verbal responses—primarily, use of their hands, with palms facing together and moving in and out from each other to indicate the size of parts (see Figure 5.10), or a cutting action that was closely connected to their description of partitioning, splitting, or sharing (see Figure 5.11).

Figure 5.10

Recreation of Gesture Used by Children

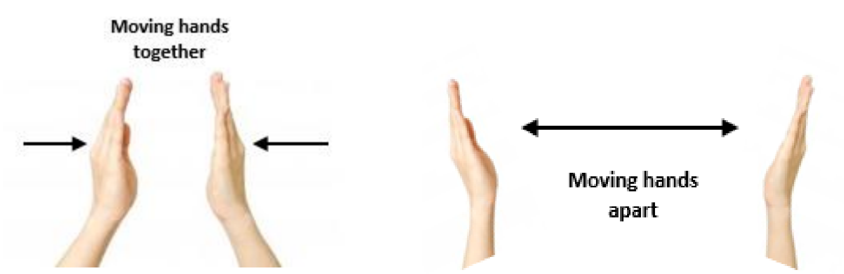


Figure 5.11

Recreation of Cutting/Sawing Gesture Used by Children

For example, Children 28 and 44 used the gesture in Figure 5.10 to describe how the pieces of a cake get smaller when there are more people to share between. This indicates the children were visualising the action of partitioning, and the relationship between the size of the part and the number of parts.

Seven children were observed specifically using gesture that suggests they were 'dealing' out fair shares in an array-like structure. Three of these children were observed and reported on by Teacher B:

> Children 43, 35 and 29, mimicked Chelsea using their hand on the desk to signal a grouping of objects—these boys [were] grouping the counters in a square or rectangle and dotting or sharing out the counters in that shape when describing fair shares. (Teacher B, written observation)

When asked about this use of gesture by Teacher B, the children all referred in some way to 'seeing' the geometric structure of different units created within the same set. For example, Child 43 cupped their hands over the counters that were grouped in a square to explain that the 'square' was one fair share—indicating the interplay between spatial structure, gesture, and the many-as-one idea.

5.4.1.3 Lesson 5: Cookie Fraction Estimation (Part 1)

In the Launch phase of Lesson 5, there was evidence that children had established the idea of equality for this key indicator. This phase of this lesson involved children exploring a fraction kit that comprised of nine 12 cm-diameter magnetic circles partitioned into halves, thirds, quarters, fifths, sixths, eighths, tenths, twelfths and an unpartitioned whole. The children were invited to identify similarities and differences between the parts and units (see Figure 5.12).

Figure 5.12

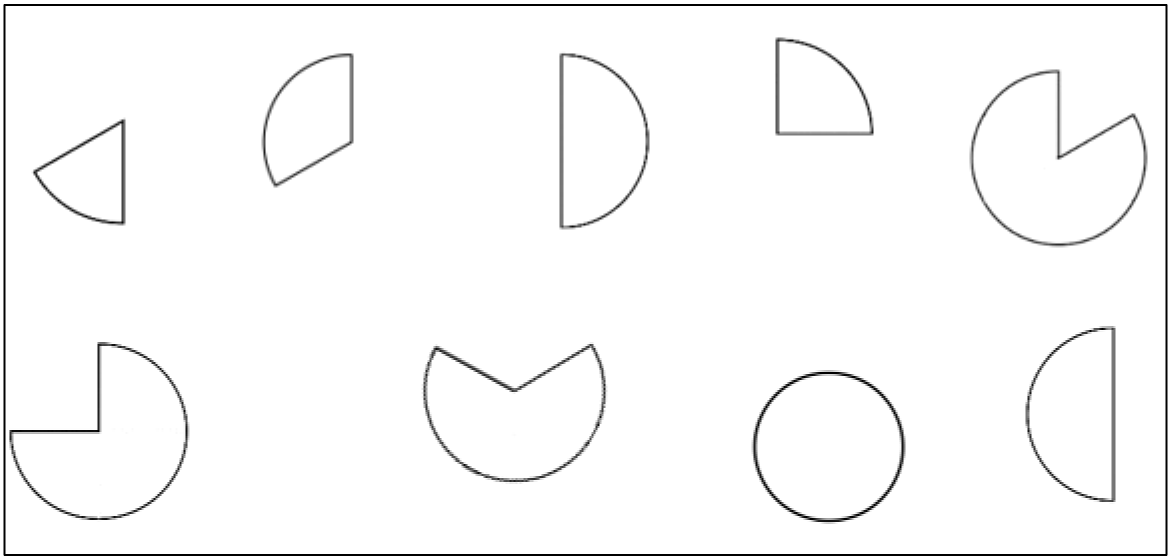
Fraction Kit Model for Lesson 5

Children were observed stacking different parts on top of each other to visually compare the magnitude of different units. Some were observed creating composite and equivalent units, such as comparing 1-half with 6-twelfths.

During the Explore phase of this lesson, the following problem was posed to the children based on the characters in the picture book The Doorbell Rang (Hutchins, 1989): ‘Victoria and Sam left the cookie jar open one afternoon, and a mouse got in! It ate some of the cookies. Can you tell how much of each cookie the mouse ate?’ The children were asked to determine how much of a cookie had been eaten by a mouse from different representations of a cookie's parts (see Figure 5.13).

Figure 5.13

Worksheet for the Cookie Fraction Estimation (Adapted from Way, 2011)



The children were also asked if they could make a whole cookie exactly by combining any of the parts. Some children described a strategy of imagining 'turning' and 'moving' the pieces to match up each pair, suggesting they engaged in spatial visualisation to create a whole from composite units. For instance, Children 47 and 49 gestured with a circular motion of their hands (like they were turning a dial) and used spatial language when describing the cookie with one-third missing. Child 47 stated:

We can see that that chunk is like a third because in our head if you move this piece [the missing third] around, you'd get another two of the thirds to cover the whole cookie.

Child 38 demonstrated a similar relationship between spatial visualisation and the use of gesture when describing how much of a cookie had been eaten based on 1-quarter being visible:

I imagined folding this part [1-quarter] over [signalling a mirror image flip with their hand] to make a half, and then, you know, the other side that makes a half is two perfect quarters—so the missing part has to be three of these quarters. This description suggested Child 38 was visualising the doubling and halving idea to identify the composite unit, suggesting an awareness of the units within units or splitting (rather than counting) basis. However, their statement suggested the composite unit idea is still developing, as the phrase ‘three of these quarters’ indicated they may not consider 3-quarters necessarily as a unit but rather as the measure, or possibly a count of individual unit fractions.

The above examples are indicative of the way the children successfully engaged with this first key indicator of the intervention. The awareness of, and attention paid to, the geometric symmetries and similarities of the 2D shapes in continuous models suggest children were visually comparing these parts and, at times, mentally moving or doubling them, suggesting spatial visualisation played a key role in the development of the early fraction as a measure and fraction as an operator understanding. Similarly, the arrangement of sets of objects, where children were inclined to make geometric structures or row-and-column arrangements to better describe and justify the equality of the parts, suggested spatial structuring was a critical element in developing this key indicator.

5.4.2 Reinitialising the Unit

The focus of the second key indicator was for children to develop an understanding of reinitialising the unit, which is the process of working with ‘unit of units’ (Confrey & Smith, 1995). It builds on the distribution and partitive division ideas explored with the fair sharing and many-as-one ideas in the previous indicator, but there is focus on being able to quantify the various units in relation to the number of partitions made. A selection of tasks in this stage of the local instruction theory will now be presented to exemplify children’s development of these ideas.

5.4.2.1 Lesson 5: Cookie Fraction Estimation (Part 2)

As described in Section 5.4.1.3, Lesson 5 was designed to develop children's awareness of composite units in continuous models through an appreciation of size of parts, evoking spatial visualisation. However, there were several examples of children reinitialising the unit through this activity.

During the second part of the Launch phase of this lesson, Child 43 provided some insights into the way they reinitialised units. They commented that when exploring the fraction kit to work out the unit fraction, they did not have to make the whole when using tenths. They first matched the part they thought the mouse ate, a tenth, and stated, 'if five of those pieces covered a half, then it is the right piece—because half of 10 is five. I just doubled and flipped it over in my head to work out the right fraction'.

Here, it is evident that Child 43 is developing the many-as-one and composite unit ideas from the fraction as a measure meaning, intertwined with the doubling and halving idea from the fraction as an operator meaning. Additionally, the description and understanding appears to be supported by the child's use of visualising and gesturing this act of doubling – suggesting the child was engaged in spatial visualisation to reinitialise 5-tenths as 1-half.

Similarly, there were other instances of children exploring and reinitialising units while paying attention to how they were partitioning. For example, Child 35 stated in their summary of this task:

> It's opposite in a way to normal numbers, you have eight people, but the groups and parts are heaps smaller because you're sharing the thing with so many [gesturing a folding-like gesture, and then palms drawn together to indicate smaller parts] than if you just had the whole thing to yourself or just two people.

This statement (along with Child 43's example) suggested Child 35 is developing their own understanding of reinitialising unit, which is an appreciation that a whole can be renamed (infinitely) as a set of units of units. Although this understanding is emergent in these examples, this is an important key indicator for children to build because it is what separates the multiplicative foundation of partitioning (Confrey & Smith, 1995) from an additive understanding.

5.4.2.2 Lesson 6: Tablecloths

In Lesson 6: Tablecloths, the children were presented with the following problem:

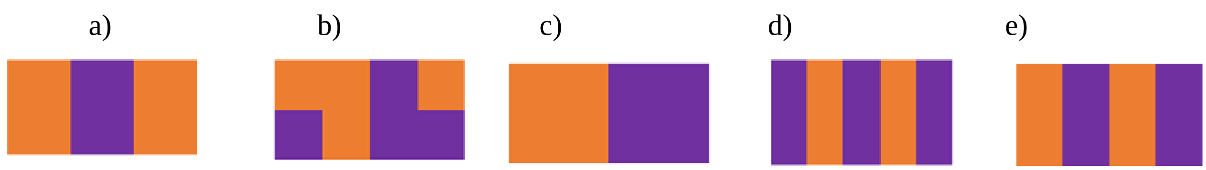
Ma wanted to buy a new tablecloth for the kitchen table. She asked Victoria and Sam to go to the shops and see if they could find one that was suitable. She asked for it to be in the colours of purple and orange, but she wanted it to be more purple than orange.

Victoria and Sam found the following tablecloths. Which of the tablecloths can Sam and Victoria choose from? How much is the purple part in each cloth? How do you know?

The children were provided with a range of A2 laminated images of the tablecloths Ma had to choose from (see Figure 5.14).

Figure 5.14

Examples of Tablecloths Presented to the Whole Class for Lesson 6



Several children (47, 32, 27, 28 and 36) described that if they 'move' different parts within a tablecloth—suggesting spatial visualisation—they could 'see' which tablecloths had the same amount of purple and orange; indicating an awareness of spatial proportional reasoning. For example, the children described how they could imagine swapping the centre orange and purple vertical segments in the middle of tablecloth ‘e’ to demonstrate it had the same proportion of colour as tablecloth ‘c’. Similarly, the children described how they imagined rotating the small orange and purple squares in tablecloth ‘b’ in a clockwise fashion to again justify the spatial proportions of each colour were the same size across tablecloths ‘b’, ‘c’ and ‘e’.

This understanding of unit and composite unit fractions as fraction as a measure ideas, is exemplified by the following response:

> Child 30: You can have something [gesturing parts of the third of the tablecloth] that has lines all over it, and all different shapes, but it’s still a whole, and you can still make a half or a fourth if you look inside these patterns and move them in your head.

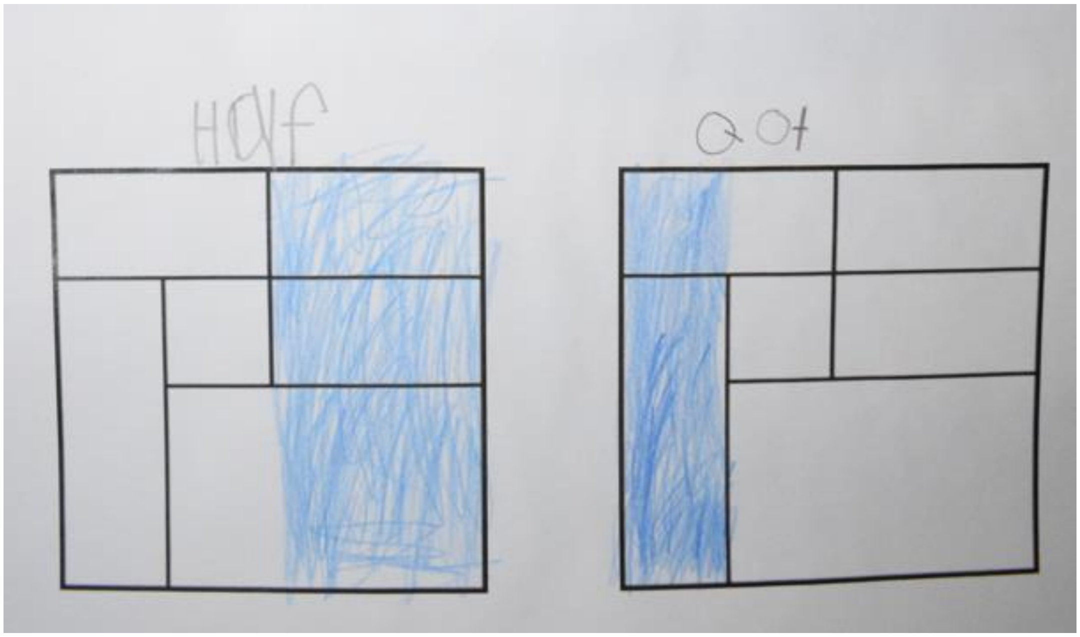
With this, Child 30 gestured to encapsulate the whole of tablecloth ‘b’ with two hands, then gestured moving the smaller orange and purple squares to the opposite sides of the rectangular area. This child thus indicated an understanding of the distribution idea from the fraction as a relation meaning, whereby they visually compared (via spatial proportional reasoning) whether a tablecloth is more or less than one colour. The interpretation of this child engaging in spatial visualisation was supported by their use of spatial transformation vocabulary; that is, words indicating the child was mentally changing parts of the image in some way – such as move, turn, and slide – were captured in conjunction with the child’s gestures that supported their explanation of how they established ideas about the proportions of colour. Additionally, prepositional terms indicating where the fractional pieces were in relation to one another were also captured (e.g., next to, above, beside, diagonal). However, these terms were not as prevalent as the use of gesture when the child was communicating their ideas.

The Explore phase of the lesson required children to create their own tablecloth with varying proportions of colour, either using a pre-patterned tablecloth or designing their own patterns within a blank rectangle. A group of three children created the representation shown in

Figure 5.15, and I questioned them as they negotiated how they would represent the unit fractions of 1-half and 1-quarter.

Figure 5.15

A Representation Created by Children 37, 29 and 52



R: What do you notice about the patterns on the tablecloths?

Child 37: It's confusing because there's all different shapes.

Child 29: Yeah, but half can just follow this line, because it's half [referring to the vertical line halfway through the tablecloth on the left tablecloth in Figure 5.15].

Child 52 coloured the right side of the tablecloth in purple, stating, 'you can make your own lines for the parts, because it's still coloured in the half of the tablecloth', indicating an awareness of spatial proportional reasoning. The reference to 'making your own lines' suggested Child 52 is creating units within units (e.g., reinitialising) in this context through visualising the partitioning operation, and the use of spatial proportional reasoning to compare the size of the parts created.

These children continued with the next tablecloth and decided to colour it 1-quarter purple. Child 52 immediately coloured in the portion of purple. When I asked, ‘How do you know this is a quarter?’, Child 52 paused, indicating they might have been following the printed lines rather than considering the coloured part in proportion to the whole. Child 37 interjected, ‘because you can see it’s two of them in the half, so four altogether’. Child 29 continued, ‘if you start with half, it’s two, then you get two more parts in each [half]’. These descriptions indicate the children were thinking and visualising partitioning from a process of repeated halving to determine the different parts, which is indicative of the fraction as an operator meaning. This is a complex task, given the distractions the pre-patterned tablecloths presented. Yet, the focus on visualising the relationship between the outcome of recursive multiplication (reassembly) seemed to assist with their understating of the size of the parts created.

I asked the children if they could describe how much of the second table was white, if a quarter was already coloured purple. Child 52 flipped their hand from palm side down to palm side up in relation to the purple part and explained, ‘there’s another 1-quarter there and then two-quarters in the other half…so you get 3-quarters [white] to a quarter [purple]’. The flipping hand gesture in conjunction with the explanation, suggests the child visualised the splitting and process, to determine the different fractional parts. They could also visually determine the relationship between halves and quarters, suggesting they are developing the composite unit idea from the fraction as a measure meaning.

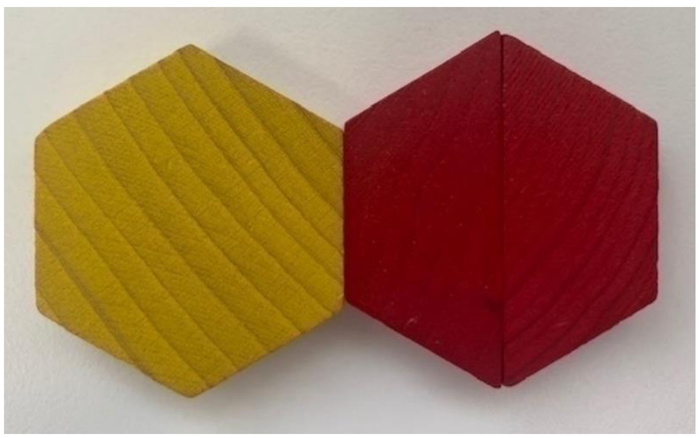
5.4.2.3 Lesson 7: Pattern Block Fractions

Lesson 7: Pattern Block Fractions required children to consider a regular shape (initially a hexagon) as a whole, and then describe what a half, third, double and other fractional amounts might be. They were also asked how to represent these fractions with the blocks. For example, they were asked, ‘If this is a whole [hexagon], what is half?’ The children were provided with a variety of pattern blocks, such as hexagons, equilateral triangles, squares, two types of rhombi and trapezoids to explore common and mixed fractions.

In this lesson, the children were asked to enlarge their pictures by double or shrink them by a half and so forth, for a focus on fraction as an operator meaning, in addition to considering how they could describe the relative quantities from fraction as a measure perspective. The arrangement in Figure 5.16 was created by Children 31 and 42.

Figure 5.16

Recreation of Children 31 and 42's Pattern Block Representation of 'One'



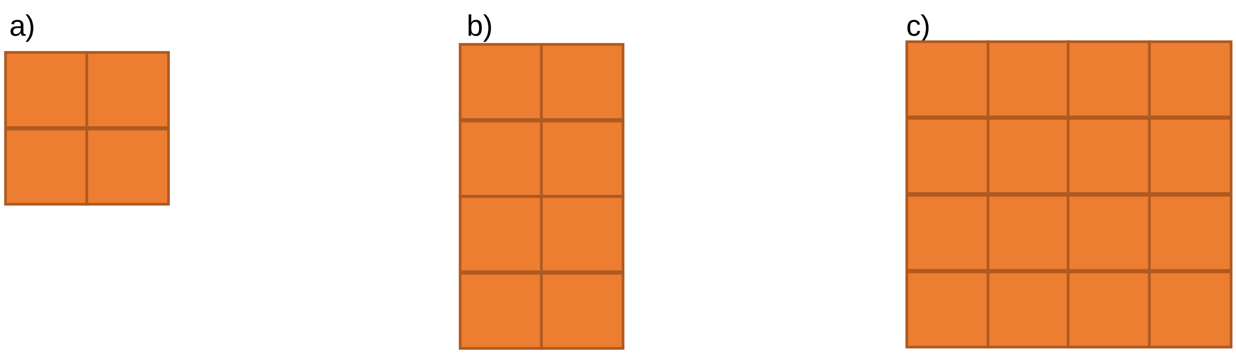
Child 31 stated they began with one hexagon and two trapezoids as 'one', because each colour proportion were halves as they were 'the same on both sides'. This suggests the child was referring to the geometric symmetry between the red and yellow areas. Child 31 stated that '[if you] look inside the shapes and see the halves of each'; therefore, they were able to identify different unit fractions. They stated that one trapezoid was 1-quarter because it was 'half of the red bit, which is half of the "one" bit' (referring to the whole representation).

This representation and explanation indicated reinitialising the unit through the fraction as an operator meaning from referring to the halving and 1-nth-of... ideas. It also revealed a close association with the many-as-one and part-whole ideas from the fraction as a measure meaning, as the child explained seeing the relationship between halves and wholes. This understanding was supported by the geometric symmetry, which is a spatial structural awareness of (and visual properties of) the blocks in this arrangement.

A common behaviour exhibited by several children in this activity, was experimenting with the doubling and halving idea with the same pattern block shape. For example, Children 34, 28 and 49 worked as a group with squares to produce the representations shown in Figure 5.17 (recreated from Teacher B's notes).

Figure 5.17

Children 34, 28 and 49's Representations Created with Square Pattern Blocks



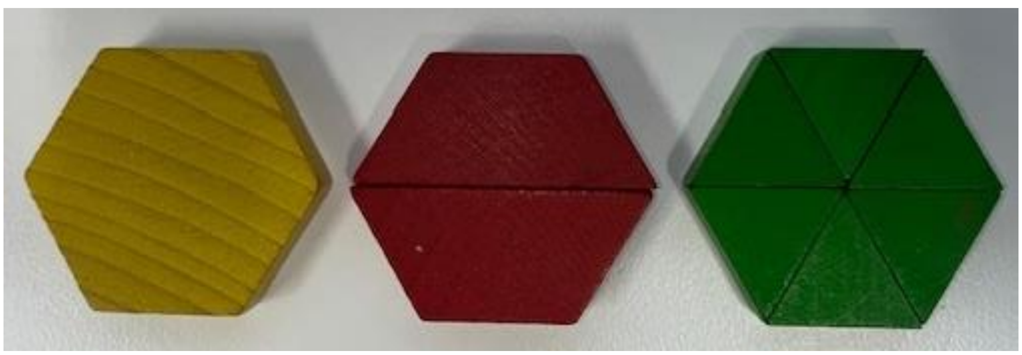
Teacher B asked these children to explain their thinking. Child 28 stated that it is a pattern of doubles (paraphrased by Teacher B). The group had used the square in Figure 5.17a (which was the unit they referred to, made of four smaller squares, which suggest they were reinitialising the unit as 1-four and 4-ones. They described that they doubled the unit of four squares each time. Child 49 described that half of Figure 5.17a is actually only '1-fourth' of Figure 5.17c because they could see the squares (unit of 1-four) inside the 'bigger squares.'

A group of children working alongside this trio joined the conversation. Child 27 described that a column of four squares is also 1-fourth (referring to Figure 5.17c) because it is the same quantity, just positioned differently. The discussion continued, with the children stating all of the ways they could represent half and quarters within each representation. The awareness of the spatial structures of the blocks appeared to support the identification of the different unit fractions and composite unit fractions, which are fraction as a measure ideas. Further, the spatial structure of the blocks enabled the measures to be determined though doubling and halving, 1-nth-of... ideas, which are derived from the fraction as an operator meanings. Therefore, spatial structuring was an important construct in reinitialising the unit for these children.

A similar experience was observed with four children using the hexagons. The children had used the hexagon, trapezoids and triangles in the representation shown in Figure 5.18.

Figure 5.18

Pattern Block Representation by Children 42, 38, 35 and 34



Note. The pattern block representation was recreated and photographed by the researcher from the reflective journal notes taken during the lesson.

I asked the children what they had created, and they replied ‘thirds, but the thirds were also in littler parts—just like in the tablecloths’ (Child 35). I asked them to explain what they meant by this. Child 34 said the representation was one whole, which they established from ‘doing three times the hexagon…so a third is one hexagon’. This description shows a relationship between the times-as-many and unit fraction ideas, which are derived from the fraction as an operator and fraction as a measure meanings, respectively. The children stated they started with three yellow hexagons, but then swapped them out to show the same regions in smaller, equal parts. During this observation, they spent time stating how they ‘could see’ smaller fractions within the hexagon—such as the trapezoids were ‘half of the one-third’ (Child 42) and ‘you can have six of these [triangles] as the same as the third’ (Child 34). However, as expected, they did not name the green triangles as eighteenths of the whole region.

The relationship between identifying 1-nth-of... and re-unitising within different pattern block representations indicated the children were visualising and experimenting with the repeated spatial structures of the various unit fractions, and their relationship between units of units, which is evidence of reinitialising.

5.4.3 Recognising Proportional Equivalence

Recognising proportional equivalence is the next key indicator of the local instruction theory. The development of proportional equivalence was promoted by the emphasis on spatial proportional reasoning. The connection between how the three different meanings of fractions were developed in relation to this key indicator is now exemplified through an analysis of Lessons 8–10 of the intervention program.

5.4.3.1 Lessons 8–10: Mapping Activities

Lessons 8–10 were based on the picture book *Knock, Knock Dinosaur* (Hart, 2014), as described in Chapter Three. Across these three lessons were various activities that required small groups of children to explore fictional town maps, and a set of instructions to locate the missing dinosaurs (see Appendix C). Throughout Lessons 8–10, the children were engaged in two main problem contexts, described below:

> Problem Context 1: *The dinosaurs have escaped the boy’s house! They’ve decided to explore the neighbourhood—here is the map. Somebody said they saw a T-Rex halfway between the boy’s house and the zoo. Where could she be?*

A range of clues were provided for the whole class to work on the same map.

Problem Context 2: *You have taken a helicopter out to see if you can find some dinosaurs. On your group's carpet map of the town, match the dinosaurs to their locations by reading the clues provided*.

Children were then asked to draw parts of their carpet map, that described where they saw each dinosaur.

These problems predominantly engaged children in fraction as a measure and fraction as an operator ideas simultaneously, by drawing on the composite units, unit fractions and part-whole ideas, to explore fractional measures of various paths. In addition, the 1-nth-of..., times-as-many, similarity, and scaling (fraction as operator) ideas were developed by children comparing how different pathways could be partitioned. This also connected to the fraction as a relation meaning when exploring ideas about distribution to compare proportionally equivalent fractions. Children were asked to spatially visualise different pathways, and use spatial proportional reasoning to estimate and justify the different measures and quantities generated. The drawing tasks required children to preserve the relationship of the fractions of pathways in scaled representations. A selection of data is now presented from each problem context.

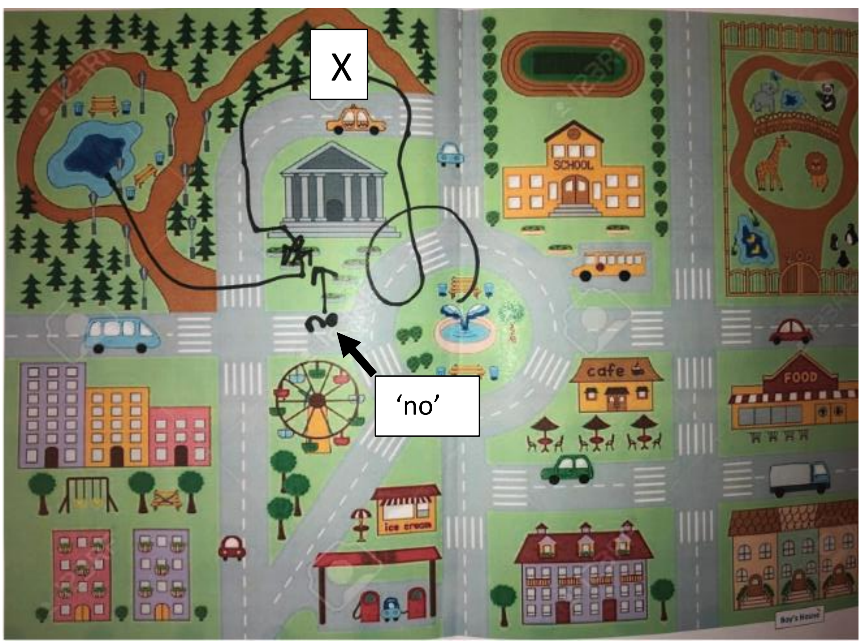
Lessons 8-10: Problem Context One

In the first example of Problem Context 1, the children were prompted with, *A T-Rex was spotted halfway between the central fountain and the duck pond—where could she be?* From observational data and work sample analysis, 18 of the children recognised the fraction as a measure meaning for this activity independently, and engaged in a spatial strategy to solve it. For example, this was indicated by children drawing lines 'as the crow flies' on the map (some gesturing the start and end points with their hands or dragging and 'walking' their fingers along the map to signify a pathway) to determine what distances needed to be partitioned between the landmarks.

Additionally, rather than drawing a straight line, eight children interpreted this task as finding the halfway point of the path the dinosaur *may* have taken from the central fountain to the duck pond, as indicated by Child 48's work sample (see Figure 5.19). That is, these eight children created inventive pathways from one landmark to another, and indicated that they used spatial reasoning to determine the halfway point.

Figure 5.19

Work Sample Created by Child 48



Note. Work sample digitally enhanced for ease of reading.

On Child 48's map, it was evident that a point on their path had been marked with 'no' (digitally annotated in Figure 5.19 for clarity). When I asked them what this meant, the child explained that they had marked a spot as halfway between the fountain and duck pond in the same place as their friend sitting beside them. This child soon realised their friend was indicating the halfway point of a *different* path than Child 48 had initially drawn. Child 48 stated that they had to 'straighten out the line [drawn path] in my head', and that when they considered the first mark, they realised this was 'more like a three-part of the way [a third] [using their hands to gesture the three parts of the path], than a two [half]'. Child 48 then placed an 'X' on the path above the yellow car as the halfway mark instead (digitally annotated in Figure 5.19 for clarity). To paraphrase, Child 48 stated that it did not matter how long the path was, to be half meant there were two equal parts of the concerning path, but you needed to 'see' (visualise) what the whole path was first (by mentally straightening, as indicated). This explanation of the use of spatial visualisation to determine the halfway measure of the path also indicated the child is exploring the key indicator of recognising proportional equivalence. That is, they recognised that each pathway could be different lengths, but they needed to 'see' the whole pathway of each before determining if the fractional measures were proportionally equivalent. This suggests they are developing flexibility in how they view fractions and their measures, as it is not fixed to a particular context (in this case, a specific path) but is indeed dependent on the individual whole (path).

Like Child 48's response, seven children indicated an understanding of the part-whole, unit fraction ideas underpinning the fraction as a measure meaning by their description of a path, they mentally partitioned. Two additional work samples are presented as typical interpretations of this task (see Figures 5.20 and 5.21).

Figure 5.20

Child 45's Work Sample

Child 45 identified the halfway point on the paths as the café, and stated they had to 'stretch' out the path to identify where halfway would be. This child explained they had to make sure the length of the path either side of the dinosaur was the same, while running their index fingers over each part like they were measuring them. This suggested the child was trying to visualise the length of each unit of half in relation to where they had located the dinosaur.

Similarly, Child 46 provided an inventive path that suggested the use of spatial visualisation supported by their use of gesture. Child 46 determined the Ferris wheel as the halfway point of a pathway that has several bends and turns (see Figure 5.21).

Figure 5.21

Child 46's Work Sample

In the child's explanation of halfway on their path, they used a gesture like they were picking up both ends of the drawn path and pulling it like a piece of string, to stretch it out straight. The child stated that this is how they were able to imagine (visualise) where the dinosaur needed to go.

This evidence suggested the children were thinking about the relationship between half in relation to the whole path and were employing spatial visualisation to imagine the length of the path if straightened. In addition, their experiences suggested they were using spatial proportional reasoning to determine and compare the proportionally equivalent measures of the pathways.

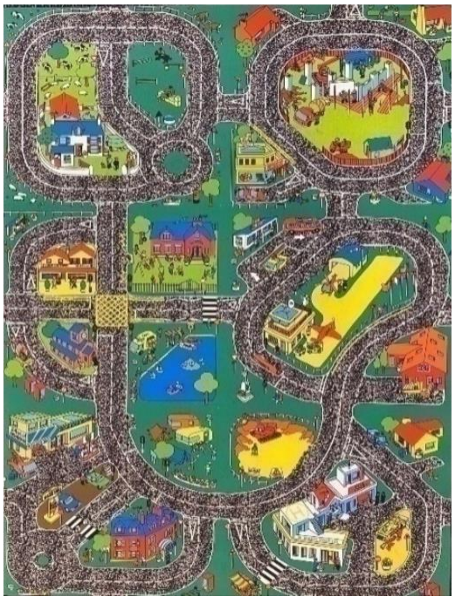
Lessons 8-10: Problem Context Two

Problem Context 2 within the suite of mapping activities involved using large carpet maps to continue 'the search' for dinosaurs (approximately 1.5 m x 1 m). An example of two of the carpet mats the children had access to is presented in Figure 5.22.

Figure 5.22

Examples of the Carpet Mats Used in Lessons 8–10

a)



b)

The children were provided with written clues specific to each map, and plastic dinosaur figurines to place on the maps when they had determined the position of each dinosaur. The clues included statements such as halfway along the carpark, three times the length of the runway and 2-thirds of the railway line. Most of these clues were open-ended, and needed some decisions to be made by the children before solving. For example, children needed to decide which end of the runway was considered ‘the start’, or whether the pathways were considered ‘as the crow flies’ before making judgments about the position of the dinosaurs. As a class, we discussed there might be more than one possibility, so the children had to clearly justify their choices.

Child 47 described how they interpreted this problem and indicated the times-as-many idea represented by their gesture: ‘If it’s three times, it is three lots of something—but it has to be the same size [using hands like they were dealing cards three times, in a row]’.

Given the class had only limited experiences with explicit teaching on multiplication at this point in the year (based on the information provided by Teacher B and current curriculum requirements) I prompted the child to tell me more. They stated:

> If you had this road [pointing to a section on the map], I'd have to measure three lots of this bit of road [gestured measuring three lengths with their hands] …if you had to find a third [of the distance the dinosaur had travelled], it's like one of these parts. You can see that three times this part, it [is] like three of the thirds. (Child 47)

Child 47's description of the times-as-many idea in relation to unit fractions, followed by the use of gesture, suggested they visualised this relationship of the measure. An interaction with Child 47 at the end of the lesson helped further interpret this child's understandings. I asked this child if there were any other ways you can have three times-as-many of something. Child 47 responded:

> If I had three lollies, and I had three times-as-many… [long pause] …I'd have to have three groups of the same lollies… so…. [bundling three groups of three fingers together representing a group, then just using one finger as representation of a multiple of three] … it would be nine.

By moving from a continuous to a discrete model to discuss their understanding of this idea, it demonstrated this child may indeed have a multiplicative awareness of this mathematical relationship. This is indicated by the way they were able to generalise their understanding of the quantities to another context, which is an example of proportional equivalence. This understanding illustrates the fraction as an operator meaning, involving the ideas of reversibility of partitioning, which was supported by visualising the relationship between times-as-many and 1-nth-of... (to identify a unit fraction) in relation to different wholes. This is another example of how multiplicative partitioning differs in contrast to a measurement, iteration-based partitioning approach to fraction understanding (Confrey & Scarano, 1995; Corley, 2013).

The understanding of proportional equivalence was also evident during the tasks that required the children to draw scaled sections of the carpet maps in their workbooks. In Lesson 9's Launch phase, children were given a thin strip of paper and asked to fold it into half and then into quarters. Using this paper as a scaffold, I then drew a line on the whiteboard and asked them to consider this as a 'path' that a dinosaur might have walked (see Figure 5.23).

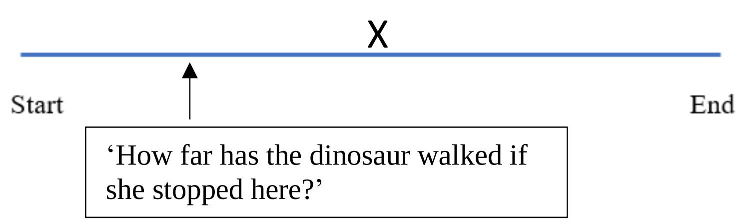
Figure 5.23

Example of a Dinosaur's Path Drawn onto the Whiteboard

Children were invited to identify where the halfway point on the path above was, using their strips as a guide. In the halfway example, all children confidently stated that the dinosaur would be halfway along the path, and a child added a cross on the path to demonstrate (see Figure 5.24). Next, I pointed to 1-quarter of the original path (indicated in Figure 5.24) and stated this was how far the second dinosaur had walked.

Figure 5.24

Reference Made to 1-Quarter of the Pathway



I asked children to describe how far the second dinosaur had walked. However, when I asked the children about the 1-quarter mark, several children said they were still *in* the first half, so they were halfway. This type of thinking suggested a familiarity with the area model, perhaps because the children were treating the strips of paper as area models rather than thinking about them as linear measurement models. This was an alternative conception also demonstrated by some children in the pilot phase. To further develop the discussion, I stated the following (paraphrased by Teacher B during this lesson).

> R: We can use this number line like a ruler to measure how far the dinosaur has walked. When we use a ruler [I held up a 30 cm wooden ruler] to say I've walked the whole 30 cm… where would I be if I started at 0?

Child 34 explained by running their finger along the ruler until the 30 cm mark. It was then considered as 30 cm they had walked. I went on to explore the area previously identified as troubling for the children:

> R: If I were to walk the whole length of this ruler, I couldn't put my finger *here* [approximately 2-thirds along] and say I had walked the whole distance of 30 cm, could I?

At this point, a few children replied with, 'oh yeah!', suggesting a potential change in understanding. Child 28 described their thinking to the class:

> If I walked halfway along this line, [running their finger along the 30 cm ruler until approximately halfway] *this section* [gesturing to a half measure by placing one hand at the end of the ruler and other at the halfway point] might be half of the ruler, *but* to say I walked halfway along this ruler, I would have to finish here [dragging their finger from the start of the ruler, and pointing to the approximate halfway point].

Another example of this thinking was displayed by Child 32:

> When you walk along a path, it's like the bit you've walked that's how big your walk is, and then wherever your home is, is how far you've got to go. You can break it up too,

> like, the shop is halfway, so I've walked the same amount to the shop, and I have the same amount to get home.

Responding to this point, Child 43 said:

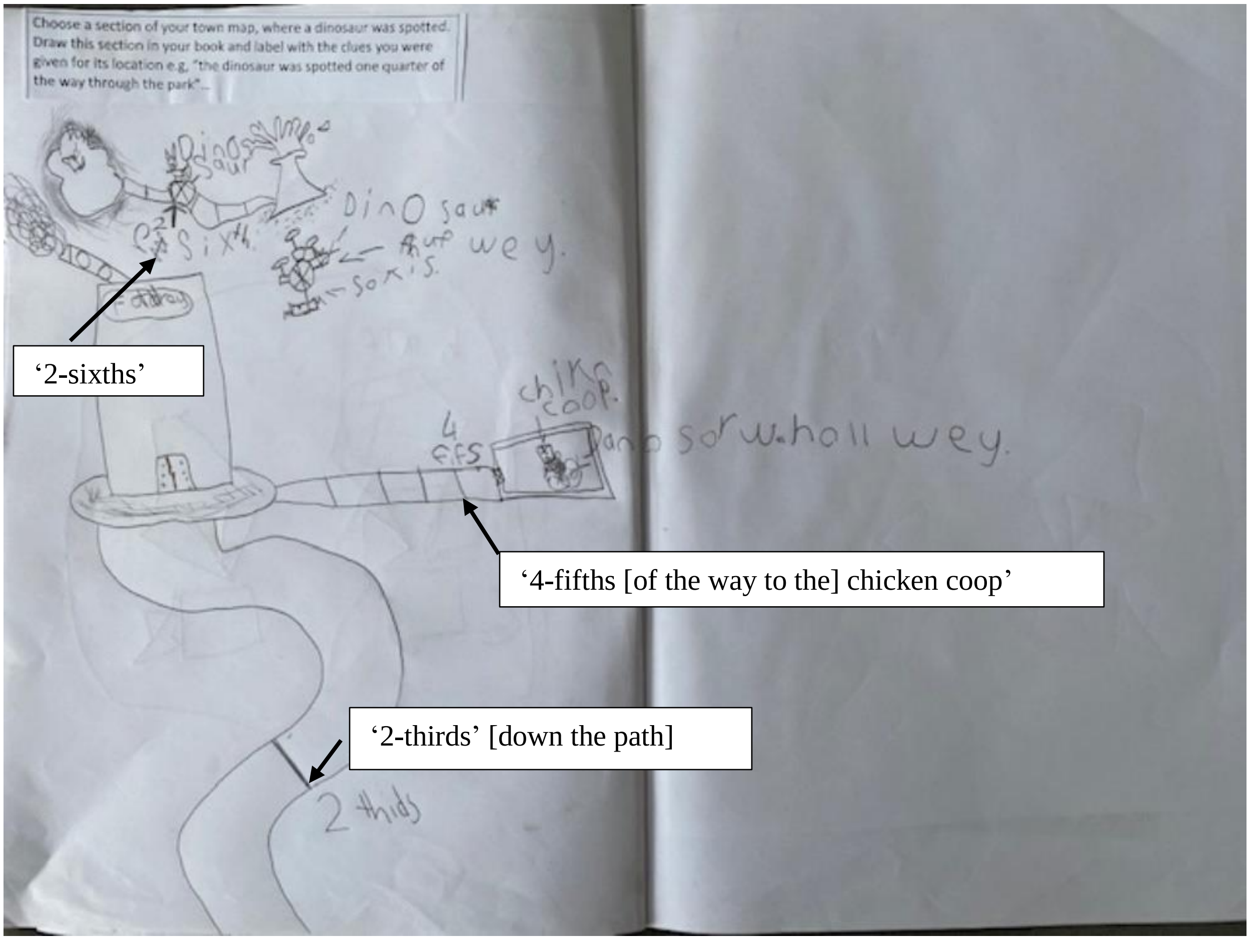
> But I could imagine that I can walk from here to far, far away, and its only half to where I'm going—like Adelaide or something. But I could walk from here to that table, and it's halfway of this room. You have to think about what the end is to know how big you've walked.

The ideas and understanding children expressed in these statements are consistent with Spinillo and Bryant's (1991) description of the 'half boundary' idea, which they argue young children develop early on, and helps build an understanding of fraction magnitude. They suggest that part-part relations, or comparing one part or half to the other part, develop before part-whole relations. In the above example, the children were originally referring to half as a discrete part in relation to its other part; however, Child 43's explanation suggested a shift to intensive quantitative understandings. This is evidence of the understanding that a half is proportionally equivalent in relation to its relevant whole (Pedersen & Bjerre, 2021). The children are reasoning about the meaning of the fraction of half, rather than seeing it as a quantity that is fixed to a particular object.

The next part of this problem context required children to draw a scaled version of part of a map with invented clues of their own, such as in the case of Child 34's description of their representation, presented in Figure 5.25.

Figure 5.25

Child 34's Representation of Lesson 10: The Dinosaurs Have Escaped (Part 3)



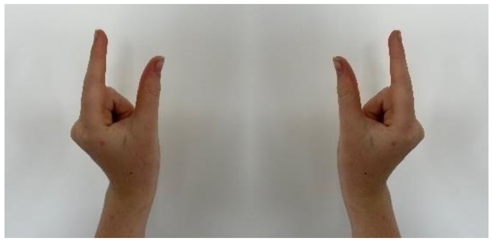
When I asked the child to explain what they had represented, they stated:

> The number of parts names the fractions, so if it's a fourth, it's four equal parts. I just thought in my head to imagine how many parts I wanted—so I wanted sixths, fifths, and thirds (Child 34).

Using a gesture similar to that in Figure 5.26, Child 34 stated they 'chunked' each path into the same size parts, after they imagined the path 'straightened out' in their mind (referring to the curved path they have marked as 2-thirds in Figure 5.25).

Figure 5.26

Recreated Images of the Gesture Used by Child 34 to Describe the 'Chunking' Operation Performed



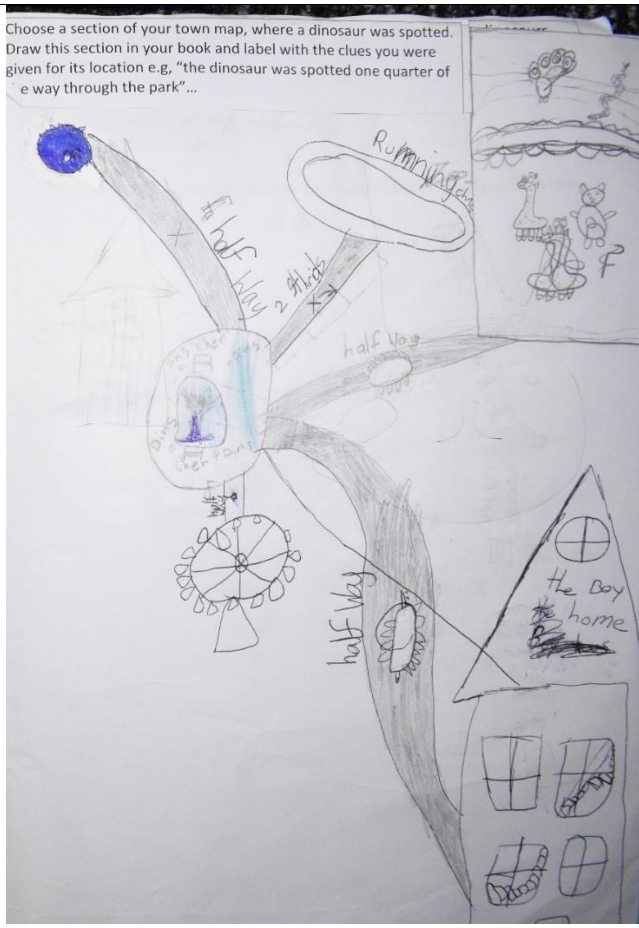
I asked Child 34 how they knew what size to make the 'chunks'. They put their index finger from each hand in the middle of the paths partitioned into sixths and fifths, in what appeared to be a deliberate benchmarking of half, while stating 'the five equal parts needed to go along this path' (referring to the pathway near the chicken coop) and 'these are smaller, sixth parts'. As they referred to the unit fractions 'chunks', they made a gesture with each of their hands, either side of the halfway point in each part (see Figure 5.26).

Although Child 34 did not explicitly articulate that the halfway points were acting as a reference for partitioning the different paths, their gesture indicated they were engaging in spatial proportional reasoning by using halfway as a guide to determine the size of the other measures. In the analysis of the classes work sample for this task, 19 children demonstrated this proportional awareness with reference to halving as a benchmark, with some additional comments provided by the children shown in Table 5.7.

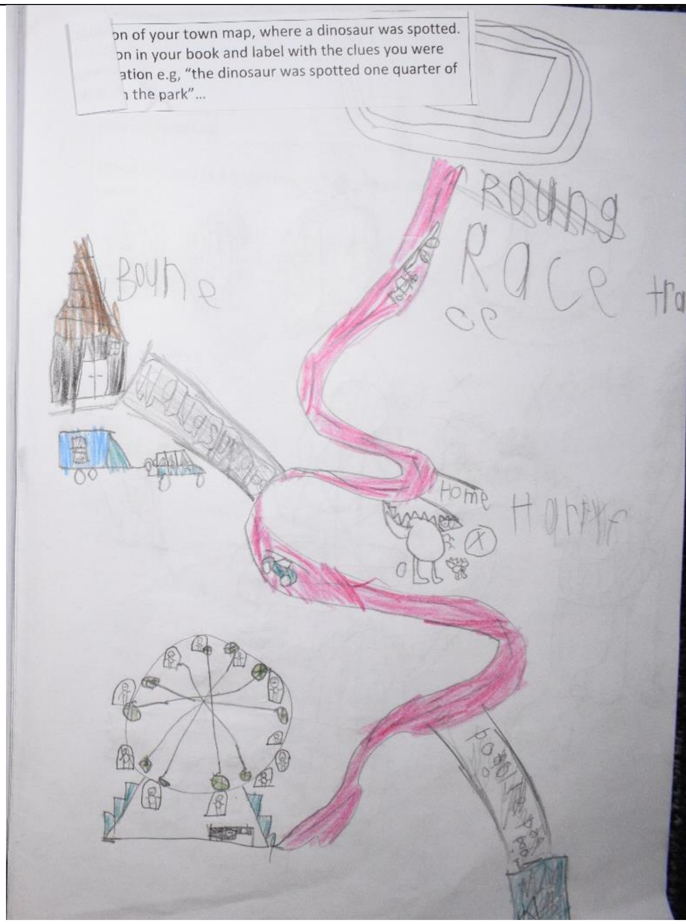
Table 5.7

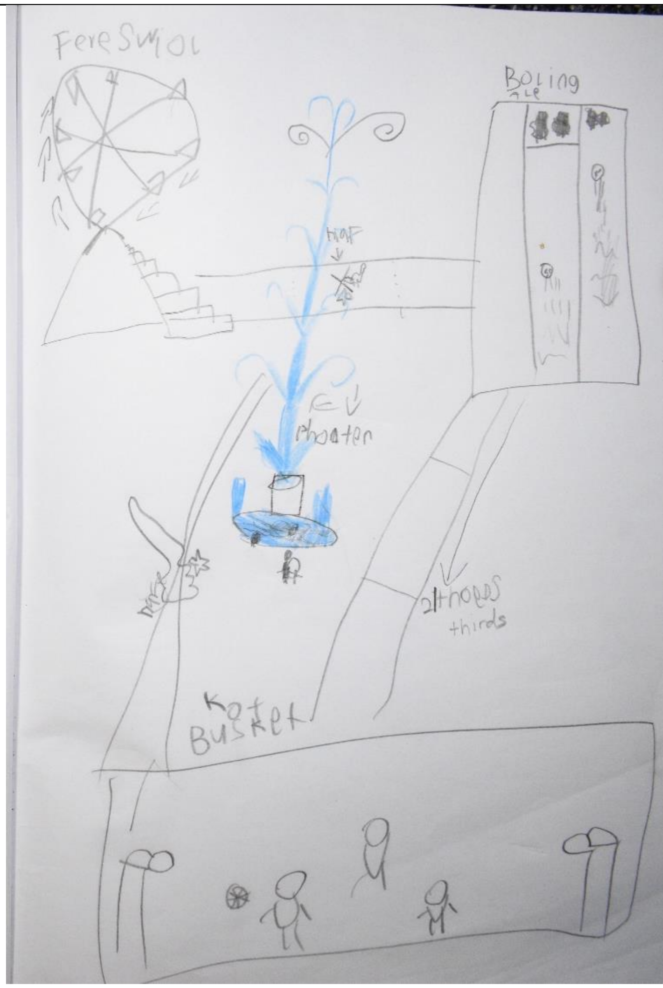
Examples of Children's Scaled Representations of the Carpet Maps

<table>
<tr><th>Child</th><th>Work sample</th><th>Accompanying observations and analysis</th></tr>
<tr><td>Child

32</td><td>
</td><td>Child 32 explained to the Teacher B that they were all different length paths, 'so half is going to look different on each, but still in the middle [of each path]'.

Teacher B also recorded that the child suggested that for the curved path, the halfway point had to take into account the length of the curves in relation to the whole. In other words, the child could not just consider halfway between the destinations as the crow flies but had to visualise the length of the entire pathway to determine the proportions of half.</td></tr>
</table>

Child 27		During a whole class discussion, Child 27 stated that they tried to make the 'windy parts' of the pink parts 'the same [length]', showing an awareness of unit equivalence in each half. I asked them how they knew the 2-halves were the same length and they said they just 'doubled the top bit over'. This suggested they were visualising the doubling and halving idea.

Child 35		Child 35 described thinking about the pathway partitioned into thirds as initially difficult. The child said they tried to 'measure with [their] fingers' where to put the thirds (indicating a counting-based, iterative approach to partitioning) but then said they knew that a third was 'a bit before half'. So, they used the half benchmark to help determine the thirds. The relationship between visualising and estimating the halfway mark to determine other fractional measures demonstrated an understanding of magnitude between thirds and half. However, this also suggested that this awareness was derived from splitting (multiplicative partitioning) approach and that their counting-based, iterative approach did not serve their purpose.

Child 45		Child 45 demonstrated a spatial proportional awareness of half in their explanation of ‘my lines are all crazy, but there’s still two same size parts’, referring to each half. With this explanation, the child used a gesture that suggested they were visualising straightening out the 2-halves, like it was a piece of string they were flattening out with the palms of their hands.

The most common gestures observed during this task were iconic in the form of a sawing action, a chunking gesture (as illustrated in Figure 5.26) and a stretching and pulling hands apart gesture when describing how they mentally straightened out the path. The use of spatial language is again evident in these contexts, specifically spatial transformations (terms that indicate movement) that appeared to support the descriptions of how children mentally visualised or ‘straightened out’ the paths. Moreover, the children’s use of spatial dimensions (which referred to the actual length or size of the parts) suggested the children’s engagement with spatial proportional reasoning, because they compared the unit fractions within a given whole, and to justify how each path was proportionally equal in comparison to one another. This is primarily connected to the fraction as a measure meaning as they seemed to be perceiving or visualising the halfway parts of different spaces. In turn, this appeared to help children determine other measures, and conceptualise the idea of fraction magnitude is in relation to its whole.

5.4.4 Connecting Multiplicative Relations

The final key indicator is developing children's ability to connect multiplicative relations of fractions and simple ratios. This means children were engaged in a range of experiences that developed their understanding of the part-whole and part-part relations of fractions, and simple ratios.

5.4.4.1 Lessons 11 and 12: Exploring Simple Ratio

Lessons 11 and 12 focused on the multiplicative, part-part foundations for ratio and fractions as a relation, such as the many-to-one idea (see Appendix C). The rationale for developing children's understanding of fraction as a relation through whole number units (such as ratio) is that it supports an understanding of fractions as multiplicative structures. This involves the coordination of units in a way that emphasises proportional reasoning (Confrey, 1994) such as the 'building up' and 'building down' (Hino & Kato 2019) strategies to work with part-to-part ratio. The selection of lessons discussed in this section are based on activities from Lessons 11 and 12. The following two problems will be discussed:

Problem Context 1: *If one dinosaur step was equal to two of your normal steps, how many of your steps would you need to take for five dinosaur steps?*

Problem Context 2: *Mum decides to feed the dinosaurs some of the pies she has made. She has made three pies for each dinosaur. How many dinosaurs will she feed with 18 pies? How many dinosaurs will eat 1-half of the pies mum baked?* (Stretch: *How many dinosaurs will eat 2-thirds of the pies*?)

Lesson 11: Problem Context One

During the Lesson 11: How many steps? activity, children were asked to represent the problem, with a range of different questions to challenge the children, such as:

- *If one dinosaur step was equal to two of your normal steps, how many of your steps would you need to take for five dinosaur steps?*
- *If we take three steps to every dinosaur step, how many steps do we take in two, three…10 dinosaur steps?*
- *What if the dinosaur had taken six steps and you had taken 18 [also combinations of three and nine; four and 12]? What is the smallest number of steps you would need to take for one dinosaur step? How can we represent this in a way that helps us describe what is happening?*

The children were provided with a bucket of chalk, invited to move outside to work and asked to think about describing the number of human steps needed for each dinosaur step. The children typically produced the representation depicted in Figure 5.27.

Figure 5.27

Children's Typical Representation of a Fraction as a Relation Problem

Note. Recreation of the children's representations carefully reconstructed by the researcher and Teacher B from our journals.

I asked a small group of children to explain why they represented their problems in this way. Children 28, 43 and 44 responded:

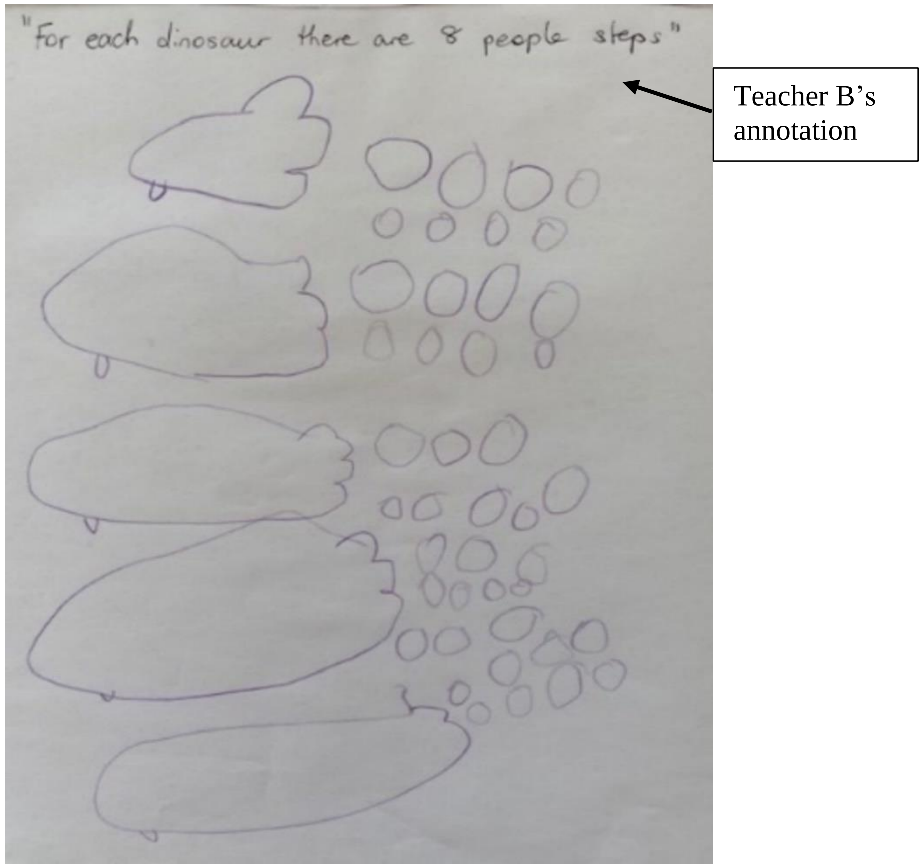
> Child 44: It was hard to think because there seemed like there were so many numbers to remember at first [referring to the overall quantities of steps in the many-to-one relationship].
>
> Child 28: We did, one [dinosaur] step, three of our steps, one [dinosaur] step, three [of our] steps…
>
> Child 43: You just keep these lines going to see the groups.

As this group of children were describing this process, they were gesturing with their hands to signify two parallel lines in relation to Child 43's comment, which corresponded with either the 'line' for dinosaur steps or for human steps, indicating the many-to-one idea. However, this also indicated a repetition of pattern and structure in the way they were gesturing and describing the relationship between the distribution of dinosaur-to-human-step ratio. This evidence suggested that children of this age can understand and explore simple ratio ideas given the right contexts. Further, it appears spatial reasoning constructs such as structure in their representations and use of gesture, were vital to them developing and communicating such ideas.

The children were asked to describe their representations, with a focus on them describing the number of human steps for each dinosaur step. They were asked to think about how they could structure the problem, so that the relationship between the dinosaur and their own steps were clear. The children were also asked to describe what they found using the following sentence structure, ‘For each (dinosaur or human) step…’. Transferring their chalk representations into their workbooks revealed that the majority of the class (19 children) displayed a structural understanding of how to coordinate units within a ratio context, exemplified by Child 33 in their work sample and description (see Figure 5.28).

Figure 5.28

Child 33's Work Sample



Note. Teacher B scribed the child's description, prompting the child to start with, '*for each dinosaur…*'.

Child 33 used an array-like pattern in the way they represented an extension of this problem—*eight human steps per dinosaur step,* indicating a structural awareness of simple ratio. This child was working with the problem, there are 'five dinosaur steps and 40 human steps, how many human steps are there to one dinosaur step?', which was posed to the child by Teacher B. Child 33 used a bucket of counters to work with the problem in concrete form, before drawing their representation above. They described to Teacher B that 'for each dinosaur [step] there are eight people steps' (however, the pictorial representation includes a total of 42 human steps incorrectly). Teacher B asked this child how they knew they were correct, to which they replied:

> For each dinosaur step there are eight of our [human] steps. So, to work out how many steps you need for 50 dinosaur steps or something… you'd just have to go counting *eight plus another eight* [plus another] *eight*, 50 times. That's a lot of steps.

The many-to-one idea is indicated by the use of 'for each dinosaur step', which I emphasised by asking the children to think about the variations of this problem as they explained their thinking. This description reveals a reference to the structure of the quantities involved in this proto-ratio, fraction as a relation problem (despite the error in human steps). Child 33 was able to demonstrate a 'building down' strategy (Hino & Kato, 2019) to determine the underlying unit ratio, which is an important step to coordinating units multiplicatively in part-part relations.

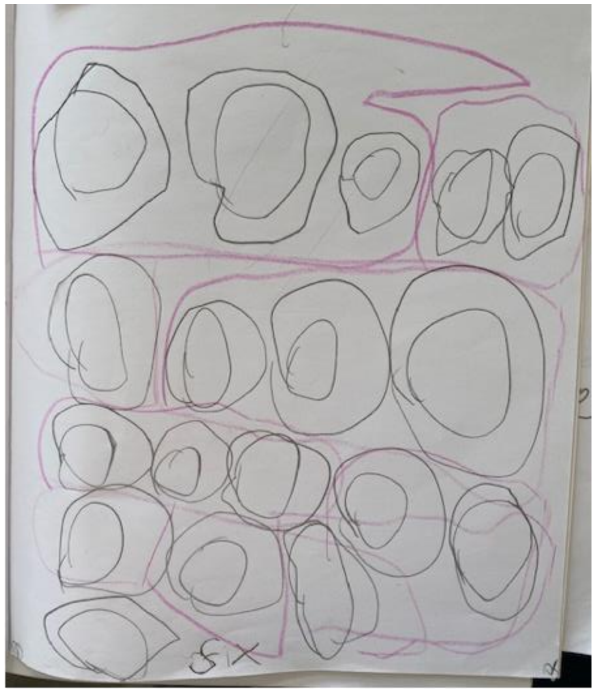
Lesson 12: Problem Context Two

Lesson 12: How many pies? required children to draw on similar ideas to explore simple ratio in the context of feeding pies to dinosaurs. The first part of the task (i.e., three pies per dinosaur, how many dinosaurs will be fed with 18 pies) proved challenging for approximately half of the class. The children who found this task difficult were unsure of how to deal with the quantities presented in the problem.

Child 46 represented a partial use of spatial structure in their initial description of their thinking and representation (see Figure 5.29).

Figure 5.29

Representation Created by Child 46



Child 46 explained their thinking to me during the lesson:

> I knew you had to have 18 [pies] to start with, so I drew them. But then, I grouped into threes next to each other, along each line so you see how many groups of pies there are—it's six [moving their hands from left to right, signifying groups being made in a linear fashion].

The explanation, along with the gesturing of the groups in a horizontal direction indicated Child 46 was thinking about the many-to-one idea from the fraction as a relation meaning by grouping three pies at a time, with a purple boundary to indicate this relationship. However, there was a trial-and-error approach observed in the way they tried to group the pies, as they made several attempts at creating equal groups in purple. The child also relied on Teacher C's scaffolding (prompting, questioning, and redirecting to the problem) to enable the representation to be created. The trial-and-error approach appeared evident among approximately half of the class (based on Teacher B's and my observations and discussions during the lesson).

In contrast there were a small number of children that provided quite sophisticated representations, and understandings of this problem. For example, Child 29 explained their process of problem-solving and representing this task, demonstrating a connection between spatial structure and how the child viewed the multiplicative relations between the fraction meanings (see Figure 5.30).

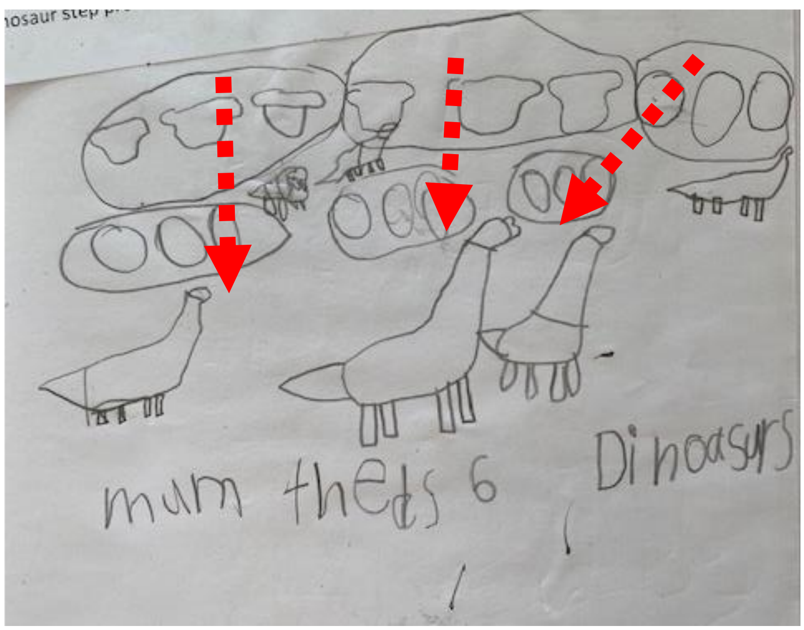
Figure 5.30

Child 29 Work sample

In this representation, Child 29 stated that rather than drawing 18 pies first and grouping them into three for each dinosaur, they drew groups of three pies, counting by three until they reached 18. When asked why they chose to draw their pies the way they did, the child stated that when you 'line groups up', they could easily 'see how much is there'. That is, the child stated they deliberately drew three groups (of pies) across the top and three across the bottom, so they could see the groups were equal. If there were more than 18 pies, the child stated they would have started the next group on a new line (underneath the two rows of pies already drawn) as threes were easy for them to count by—even though they were observed counting each pie individually. This is evidence of structural awareness in the equality and repetition of units needed when exploring the many-to-one idea, for the child indicated an understanding of the relationship between the whole set of 18 pies, and the distribution of groups of three pies for each dinosaur. The child later explained to Teacher B that two dinosaurs would eat 1-third of the 18 pies, gesturing a vertical line over their representation to indicate where they could ‘see’ thirds (see Figure 5.31).

Figure 5.31

The Movement of the Child’ 29s Finger to Describe Thirds

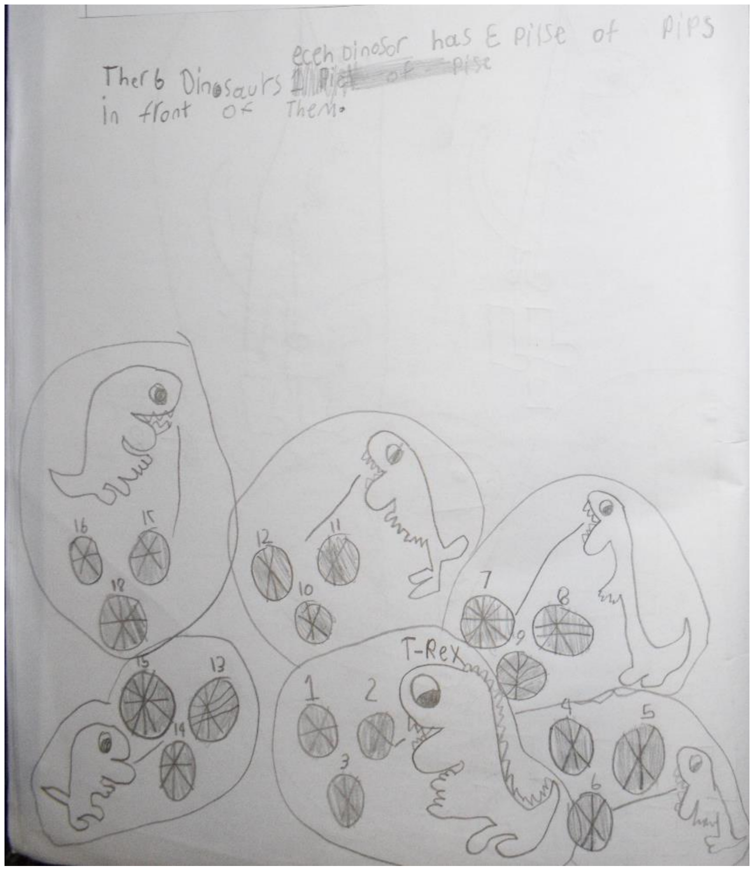


This response illustrated a connection to the fraction as a measure meaning, in that the child is now naming 18 as the referent unit; therefore, 1-third of the pies is two dinosaurs’ share, showing a connection to the composite unit idea. It is this type of response again demonstrates how the different fraction meanings are interrelated and closely connected, which is an important consideration in the teaching of this complex area of mathematics.

Similarly, Child 42 shared the following representation during the Summarise phase of the lesson (Figure 5.32), which demonstrated sophistication of understanding.

Figure 5.32

Child 42 Work sample



I asked this child to explain how they created their drawing and solved the problem. They said, ‘I just knew to draw three pies for each dinosaur, until the 18 pies are gone, then that’s how many dinosaurs there are’ (Child 42). I prompted the child to tell me about the numbers they had written next to each pie and asked if that helped them in any way. Child 42 said that it helped them know that they had ‘used up’ 18 pies. This child’s description of three pies for each dinosaur is evidence of the many-to-one idea, as a ‘building up’ strategy (Hino & Kato, 2019). That is, the child demonstrated a structural awareness and coordination of the units, where the child build ups the two quantities (dinosaur: pies) to reach the target quantity of pies (18). This is evidence of the fraction as a relation meaning.

5.4.4.2 Revisiting Lesson 1: Child 47's Discussion

While packing up to leave after Lesson 8, Child 47 asked if they could show me something they had been thinking about since the first lesson. The analysis of this interaction is included here as it is evidence of both the reinitialising the unit, and the connecting multiplicative relations key indicators.

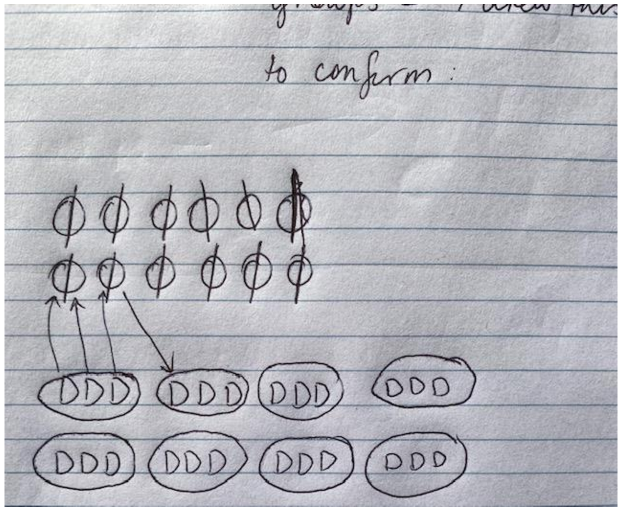
The problem they wanted to share with me was from Lesson 1: *Share 12 cookies between eight children. How much does each child receive?* Child 47 asked me if they could describe to me their solution, as they struggled to make sense of the different partitioning contexts at the beginning of the intervention. Child 47 explained:

> You could split all the cookies in half and then each person gets three of the halves, or you can just leave the one [whole] and split this one to share [referring to a single cookie drawn in their book for the first attempt]—it's the same amount. I can see in my head how you just cut all the cookies in half, and then I move them around to put them in groups of three [halves]—like, all lined up. I don't even need to write it down; I just do it in my head!

I drew an example in my notebook, instructed by the child at their request, as they had difficulty with writing and drawing representations. The child asked me to draw 12 cookies, and I asked them how they wanted me to arrange them. The child immediately replied with, 'it should be six on top and six on the bottom', gesturing two horizontal rows with their finger across the page (see Figure 5.33).

Figure 5.33

Representation Drawn from Child 47's Instructions



The child stated that to share equally between eight people, there would be four 'left over' if one cookie was initially dealt to each person from the set of 12 (which is the partitive division idea from the fraction as an operator meaning).

> Child 47: I know you can get fair shares using all the cookies if they [the cookies] are broken in half. [They gestured with their finger a line down the middle of each cookie, and I drew it on to confirm.] I line them up in groups, so its eight groups—there's eight people, and you just hand out the halves until they're all used up. There are no leftovers then!

I started to draw the groups of three-halves, and the child instructed me to put arrows indicating where the halves came from (see Figure 5.33) and then a ring around them to highlight the many-as-one unit of 1-and-a-half cookies/3-halves per person. As I was doing so, the child stated, 'It's the same as ... just three, whole cookies between two people [used hands in a cupping action to gesture a unit/group]' (see Figure 5.34).

Figure 5.34

Example of Child 47's Cupping Gesture

Given the child's statement indicating they had reinitialised the unit of 1 and a half is 3-halves, and indicated early ratio understandings of one person: 3-halves, two people: three cookies, I decided to push this child's thinking further to reveal their understanding of part-part relationships:

> Researcher: What if you regrouped the cookies again, and there were four children? How many cookies would you need if everyone still received 1-and-a-half cookies?
>
> Child 47: That's easy—it's six! Because when you go up this side [child used a gesture like their hands were a balance scale; see Figure 5.35] by that amount [double the number of children], you need to here as well [double the number of cookies]. But everyone still gets the same amount [of cookies]—one and a half.

Figure 5.35

Recreated Image of the Gesture Used by Child 47 to Represent a Balance Scale Gesture

This response suggests that child was engaging in spatial visualisation to explain how they perceived the inverse relationship between the ratio of 2:3 and 4:6. The child used gesture to signify that doubling the number of children doubles the number of cookies, which relates to the fraction as an operator meaning. Additionally, the use of gesture in Child 47’s response to my second question suggests an understanding of the proto-ratio and distribution ideas from the fraction as a relation meaning. This was indicated by the way in which the child was representing the preservation of the multiplicative relationship between the number of children, and number of cookies in the problem (Confrey et al., 2014b). This child appeared to have an advanced structural, two-dimensional understanding of the quantities (many items per child) rather than considering it as a single dimension problem (a focus only on the group of many objects or many-as-one). As described earlier in this thesis, several researchers suggest that the use of gesture provides a window into the thinking processes and visual imagery children use when considering mathematical problems (Alibali, 2005; Beilstein, 2019; Edwards, 2009). The gesture used by Child 47 suggested they were visualising the doubling and distribution of the two quantities simultaneously. This indicates how spatial visualisation supported the fraction as an operator and fraction as a relation meaning to establish early multiplicative relations of fractions.

5.5 Insights From Post Task-Based Interview

This section provides a comparison between the children’s responses in pre- and post-intervention TBIs to determine what effects, if any, the intervention program as a manifestation of the local instruction theory had on the children’s understanding of an extended range of fraction ideas. To establish the extent to which the teaching intervention influenced children’s understanding of fractions, a paired sample sign test was conducted on each set of the TBI.

5.5.1 Comparison of Pre- and Post-Task-Based Interview Responses

As described in Chapter Three, the pairwise outcomes for each item in each of Set One and Set Two were combined for the purposes of the analysis. For Class B (23 children) and Set

One (four items), this resulted in 23 x 4 = 92 possible responses. For Set Two, this resulted in 23 x 5 = 115 possible responses.

The results from the paired sample sign test for Set One and Two are presented in Table 5.8. The results indicate statistically significant positive change from the pre- to post-intervention TBI for both sets.

Table 5.8

Paired Sample Sign Test Analysis of Set One and Two (Class B)

Paired Sample Sign Test Analysis: ($p \leq 0.33$)				
Category	Assessment item(s)	Positive change +	Negative change –	*p*-value
Trusting the Count	1, 2, 3, 4	40	52	0.016
Place Value	5, 6, 7, 8, 9	58	58	<0.000

As described in Chapter Three, the pairwise outcomes for Set Three were grouped into fraction meanings and spatial reasoning categories. Table 5.9 presents the results of the paired sample sign test by category for Set Three.

Table 5.9

Paired Sample Sign Test Analysis of Set Three (Class B)

Paired Sample Sign Test Analysis: ($p \leq 0.33$)				
Category	Assessment item(s)	Positive change +	Negative change –	*p*-value
Fraction as a measure	10, 11, 12, 15, 16, 20 21 22, 24	145	62	<0.000
Fraction as an operator	10, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24	182	117	<0.000
Fraction as a relation	11, 13, 14, 23	64	28	<0.000
Spatial Visualisation	10, 12, 15, 16, 17, 18, 20, 21, 22, 24	151	79	<0.000
Spatial Proportional Reasoning	8, 11, 13, 19, 23	64	51	<0.000

Significant growth from pre- to post-test is evident across all fraction and spatial reasoning categories, with children performing well above chance. Given that the children received no other mathematics instruction during the intervention period, these results indicate that a spatialised, partitioning based approach to an extended rage of fraction meanings can substantially influence children's fraction and whole number understandings. While a useful measure, the quantitative analysis provides only a snapshot of the intervention's success, as the aim of DBR is not just to improve the outcomes of learning but to understand how the mechanisms for learning and reasoning can be supported (Cobb & Gravemeijer, 2014). To this point, examining the children's mathematical reasoning and behaviour in the post-TBI provides critical information as to how children's thinking has changed as a result of the intervention.

5.5.2 Post-Task-Based Interview Insights

In addition to the paired sample sign test, thematic analysis was conducted on the children's work samples and observations taken during the post-intervention TBI.

5.5.2.1 Set One and Two Qualitative TBI Insights

Thematic analysis of the post-intervention TBI data revealed insights in the way children engaged with the items compared to the pre-intervention TBI. In Set One and Two of the TBI, the most notable difference in the children's behaviour was their development of whole number ideas, such as composite units and an awareness of magnitude, which seemed to be promoted through the awareness of spatial structures. This section will discuss this theme in relation to Set One and Two.

An example of this theme is represented by Child 32's response correctly describing this place value idea in Item Five (26 counters). They stated that the '6' in 26 were singles or ones, where the '2' represented two tens— 'not "2" like the "6" [represents]'. As they were describing this idea, they made a gesture with their finger, of two arc-like shapes in a left-to-right motion across the table, illustrated in Figure 5.36.

Figure 5.36

Example of Child 32's Gesturing During TBI Item 5: 26 Counters

I asked Child 32 what their gesture meant. They replied:

> It's like, the '2' is a 20, not a '2' as in two of these [counters], so when you see that number, you think, '10' [gesturing first arc] … '20' [gesturing second arc] … so it's the two groups of 10, and then the six is … a bit more than halfway to 30 [gesturing a smaller, third arc].

This statement indicated the child had established an abstract composite unit (Cobb, 1995) understanding, as Child 32 considered 'ten' as one 10 and 10 ones simultaneously. It also suggested Child 32 was demonstrating spatial proportional reasoning in the way they considered the magnitude of six in relation to the next 10, and in the way they described and gestured the size of arc as being 'a bit more than halfway' to the next 10 when representing 26.

Child 32 took the 26 counters from this problem and made an array of two groups of 10, stating:

> You can see 26 is two groups of 10 and then six [more]…but is also two groups of 13—I just saw that! [suggesting they visualised this arrangement of adding two groups of three to each group of 10 to get the same result]

This example suggests Child 32 has developed an advanced structural understanding of countable units through reinitialising the unit, evidenced by their ability to make flexible statements about the different underlying units they described as 26. They were able to recognise familiar structures—in this case, the composite units of 10, six and then 13—in the way they structured the concrete materials. Reference to the composite unit of 10 (typically numerical composite of 10 ones rather than an abstract composite of one 10) and array structures (e.g., 5 x 5 counters plus one for 26) were evident in 10 children's responses, compared to just four children in the pre-intervention TBI.

Of the 23 children in Class B, nine indicated they were using spatial structure when describing elements of whole number knowledge. Some made comments that referred to using the nearest 10 as a mental benchmark when combining two collections (such as seven and nine). In addition, some suggested that 1-ten more, 2- tens less mean 'skipping' along in chunks of 10 as Child 32 stated. Moreover, seven children used gesture in a fashion similar to the example in Figure 5.36 to accompany their descriptions, suggesting that they could 'see' the structural pattern of tens as a unit.

Also notable about children's use of spatial structure in these assessment items was the accuracy in number line estimation they displayed between the pre- and post-TBIs. Item 8 (Proportional number line) required children to place the numbers 8 and 16 on an open 0–20 number line. In the pre-intervention TBI, the majority of the children either guessed or tried to count some form of invented or imagined partitions to place eight and 16 on the number line. Conversely, in the post-intervention TBI, 14 children demonstrated the use of spatial structuring and spatial proportional reasoning to accurately place eight and 16, with the majority of these children also able to attempt placing 48, 67 and 26 on a 0–100 blank number line, with similar strategies and accuracy. For example, Child 39 used their hands to partition the 0–20 number line in half, stating that '8' needed to be placed just below (on the left) of this benchmark, because it was close to, 'but not right next to', 10. They then used a similar gesture to partition the upper half of the number line in half again, stating that this represented halfway between 10 and 20—or 15, so that ‘16’ could be placed immediately to the right of this benchmark. The child commented that 48 was near the half of 50 (referring to 50 as being half of the 0–100 number line). Child 39 said they thought about the chunks that could help them, like chunking the line into fourths to then place the other numbers.

Spatial proportional reasoning was evident in the way seven children commented on the way they compared the two number lines. For example, both the 0–20 and 0–100 number lines were the same absolute length; however, the children only got to view and attempt the 0–100 number line if they demonstrated some form of understanding and accuracy with the 0–20 number line.

In the pre-intervention TBI, 11 children did not view the 0–100 number line because of their inability to complete the 0–20 task. Of those that did in the pre-assessment (n=12), five stated that they could not place one or more of the required numbers on the line because there was ‘not enough room’. The post-intervention TBI data for this item revealed that spatial proportional reasoning was evident in the majority of the 21 children’s responses who completed both number line tasks. The following responses are examples of this shift in understanding.

> Child 29: Half is like the same length on the line, but the numbers are different because there’s more to fit in.
>
> Child 42: You just think half of 100 instead of half of 20 to work out where the numbers go, even though the line looks the same [length].
>
> Child 44: It’s like half of something can look the same [running finger along number lines] but you have to think about what the whole total is, like half is 10 [pointing to 0–20 number line], but [half] is 50 here [pointing to 0–100 number line] but they are both the same [length].

This thinking indicates a development of understanding between multiplicative relations, fractions and proportional reasoning that is underpinned by both mathematical and spatial constructs (Hino & Kato, 2019). This has resulted in an increased awareness of children's whole number magnitude.

Moreover, four children used transformation terms like 'stretching out' or 'folding over and over' when making comparisons about the where the numbers should go and how the number lines were the same absolute length but represented different measures, indicating how spatial visualisation played a key role in their conceptualisation of the problem. Child 34 exemplified this use of such terms:

> Like the 20 [number line] I think the…100 is like folding it halfway, then over [halving again] so I get my quarters. And these numbers are 20…25, 50 in the middle. Then I know where I can put the 26 and 48 because they are close to these [benchmarks].

This suggested that this child used spatial structure to visualise the magnitude of the number line quantities.

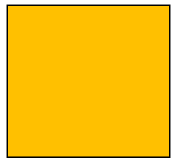
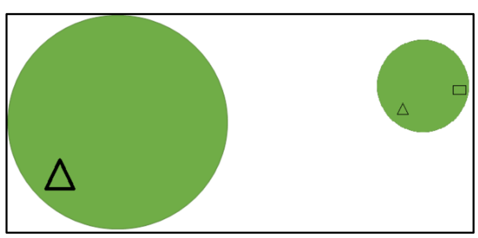
5.5.2.2 Set Three Qualitative TBI Insights

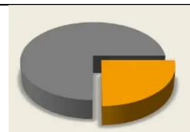
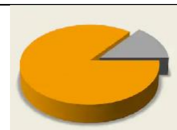
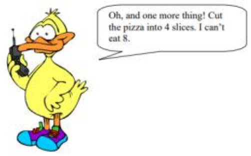
From the thematic perspective of the post-assessment data, there were patterns of behaviour evident in relation to the children's improved performance in Set Three. Specifically, the interplay between children's gesture their descriptions of their thinking suggested spatial visualisation and awareness of spatial and mathematical structures assisted their understanding.

This section will provide evidence from seven items to highlight the use of gesture and its suggested connection to spatial visualisation and structure (see Table 5.10). Although there were large improvements in many items within this set, these items have been chosen as they represent the pattern of behaviour in tasks that are quite complex for this age group.

Table 5.10

A Sample of TBI Items

Item	Description	
10 Folding Fractions	Image of a square: 	Child is shown the image of a square. *'How many ways can you imagine folding a square in half? Can you describe what you think it would look like if you folded it in half, then in half again? What is each part called?'* (University of Cambridge, 1997-2023)
11 What fraction is green?	Image of rectangle: 	Child is shown the rectangle and asked, '*What fraction of this rectangle is shaded green? How did you work that out?'* (Created by researcher)
15 Missing faces	Set image: 	Each child is shown the set and asked, *'If this group represents 2-thirds of a set, what is missing? Draw or explain your thinking.'* (Created by researcher)
19 Scale the picture	Spatial scaling image: 	An image of two circles is presented. The diameter of the smaller circle is half that of the larger circle. The smaller circle contains two shapes, a triangle and a rectangle. The larger circle only includes the triangle drawn to scale. The children are required to draw the missing rectangle in the larger circle and describe the difference in size between the two shapes. *'Can you complete the picture of the circle on the left so that it has the same shapes as the circle on the right? Explain why you chose to draw the shapes in that way*.' (Adapted from Frick & Möhring, 2016)
21 Eating pies	Images of pies:	Children are presented with a 3D image of two 'pies' that have a fractional part missing. They need to use their visualisation skills to determine what fraction each part represents. *'Can you*

	 	*describe what size each piece of pie is? How did you work that out?'* (Adapted from Way, 2011).
22 Fred's pizza	Image of Fred ordering pizza. No image of an actual pizza supplied; however, children have access to pencils and paper for use. 	*'Fred orders a pizza which he will eat the whole thing in one sitting. But…he asks for it to be cut into quarters, not eighths, because he can't eat eight slices of pizza. Does his request make sense? Why or why not?'* (Adapted from Dole, 1999)
23 Plant growth rate	Image provided of the plants as a distractor:  	Comparing proportional rate of growth. *'If plant A grows 5 cm in half a year, and plant B grows 8 cm in a whole year, which is growing faster? How do you know?'* (Adapted from Dole et al., 2012)

In Item 10 (Folding fractions), Item 11 (What fraction is green?) and Item 21 (Eating pies), the children were provided with images to prompt their thinking (i.e., a square that represented a piece of paper they had to imagine folding/partitioning, a green-and-white coloured rectangle, or a pie with a fractional part cut). Of the correct answers for these three items combined (50 of a possible 69), there were 30 instances of gesture coded in conjunction with the children's explanation. For example, a typical gesture for Items 10 and 11 were children using their hands to replicate folding a piece of paper, or in Item 11, by flipping their hands from palm up to palm down when explaining how they came to their conclusion about how much of the rectangle was green. Child 42 stated for Item 11, 'I flipped the white bits over and it wasn't probably half [of the green] it was more like…a third'. Child 29 commented that they 'flipped over the piece of pie in Item 21 to see if it was half or how many pieces would make the whole thing', using a similar gesture with their hand to coincide with describing the unit fraction and iteration of the units. This suggested the child was engaging with spatial visualisation in the problem-solving process that helped them determine the size of the parts in the representation.

Other common instances of gesture with this problem and in Item 11 were children using their hand like they were rotating parts to mentally compare the size of the fraction to its complement parts. The types of gesture were assumed to be closely related to the multiple, mental transformations the children were performing when using spatial visualisation in this item, based on the accompanying descriptions. However, this observation also suggest that children were, again, drawn to the halfway benchmark in many of these problems to determine the size, proportion, or equivalent measure associated with these tasks.

In Item 12 (Comparing unit fractions), the children were asked to explain which fraction is larger, 1-third or 1-eighth, and were not provided with any accompanying representations (although they had access to concrete materials, pencil, and paper to use as they wished). No child answered this item correctly in the pre-intervention TBI. In the post-intervention TBI, 19 of the 23 children in Class B identified 1-third as the larger unit fraction and provided rich descriptions supporting their answer that included gesture, evidence of spatial visualisation and the use of materials to justify their understanding of unit fraction magnitude. This is illustrated by the following response:

> Child 27: It's a third, because look—if have this square paper [A4 rectangular sheet] and I imagine, like cutting in this way [gesturing cutting the paper across two evenly spaced places, horizontally across the page], I get threes, each of these are a third. To get eight, you have to make more cuts and get more pieces, but the pieces get smaller and there's more of them, but they're heaps smaller—I can see them shrink. And it doesn't matter what size paper you use—a three [third] is always bigger than an eighth.

Child 27's response demonstrated an understanding of partitioning and how they were able to visualise this process using a piece of paper, in addition to using gesture to support their claims. Although there are some ambiguities in naming fractions (i.e., referring to a 'three' instead of a third), the child's reference to a third always being larger than an eighth within the relevant whole also indicates the child's understanding of the inverse relationship between the divisor and quotient—a fundamental idea in the conceptual development of partitioning. This child also demonstrated spatial visualisation through their description of the units 'shrinking' as they apply more partitions, reinitialising the unit and recognising proportional equivalence—both of which are multiplicative concepts. This is an indication that spatial reasoning skills, together with the use of gesture, appeared to assist this child to communicate their understanding of fraction magnitude, suggesting a multiplicative foundation to partitioning.

Several more children provided similar explanations:

> Child 39: It's a third. When I see the parts in my head, I imagine a line and I can break it up evenly. Just…it's like… it's the more pieces or groups [of things] you need to make out of something, the smaller they get or less you have [gesturing the forming of parts with hands, moving imagined objects to imagined groups in the air].
>
> Child 36: When you have thirds, its only three groups, so you get more in each. But I have to take some out of each three…third and move them to make eight even groups for eighths. The parts get smaller because you've moved some bits from breaking up the thirds [hands moving on the desk as the child referred to redistributing and moving parts].
>
> Child 35: You have to keep halving to get eight pieces—like folding over and over [hands moved to suggest they were folding a piece of paper], and as you do that they keep getting smaller and smaller all next to each other. Thirds are three big parts, not eight small ones.

The responses highlight how children visualised the outcome of partitioning and was able to generalise the relationship between the parts generated and their relative size. That is, the responses indicated a description of continuous contexts—such as a line that they mentally partition into thirds, a piece of paper that is ‘folded over and over’—yet also indicated a transfer of partitioning knowledge to discrete contexts, by visualising the unit fraction with a set representation. This transfer from continuous to discrete contexts is said to be an important landmark in early fraction understanding (Confrey & Maloney, 2010), because the learner is starting to abstract the idea across multiple contexts.

In the context of Items 15 and 19, spatial visualisation, structure and gesture were also strongly connected. Of the combined 46 possible ‘correct’ responses to Items 15 (Missing faces) and 19 (Scale the picture), there were 30 correct responses in the post-intervention TBI in comparison to just four in the pre-intervention TBI. However, what was most noticeable in the children’s responses to these items in the post assessment phase was the use of gesture accompanying the children’s descriptions. For example, gesture was typically associated with demonstrating how the structure of the unit fractions and many-as-one understandings of a set model in Item 15 was determined, by many children cupping their hand over each group of two faces when describing each part as a third. This also connects to the strong association of the structure of the units as discussed earlier in this chapter, again, with children in this item typically showing a structural understanding of the underlying unit formation. Two children turned their paper 90 degrees, stating a third is two faces but 1-half (of the whole) is three while using their hand to cover the units identified to model the different size unit fractions. This suggested the children have developed fluency with unitising the structure of small collections, which is a fundamental concept that underpins all of the meanings of fractions.

Moreover, for Item 19 (Scale the picture), where the children were asked to draw the missing part of an image that had been enlarged and describe how the image hand changed, a common use of gesture was children positioning their hands with their palms facing together and then moving the palms away from each other, to indicate they were enlarging the original image. Serval children used a similar gesture with a thumb and fore finger, just like one would do if zooming in on a digital touch screen (e.g., a smartphone). This accompanied many descriptions; terms such as half as big, twice as big, and double as big were typical descriptions of the relationship between the two images—which was not evident in the pre-intervention TBI.

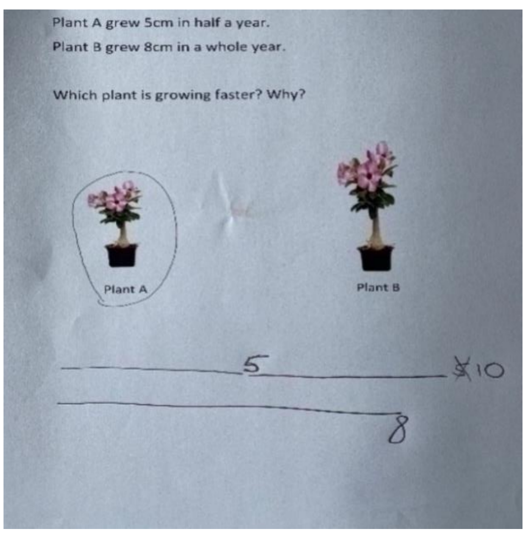
Item 22 (Fred's pizza) also showed dramatic growth in the number of children correctly responding to this item in the post-intervention (from no children pre-assessment to 12 correct answers post-assessment) and the use of spatial dimension and spatial transformation language used to support their communication of partitioning and equivalence. The problem posed was that Fred ordered a pizza that he intends to eat in one sitting, but he asks the pizzamaker to cut the pizza into four pieces only, not eight, because he cannot eat eight slices. The children are asked if Fred's request makes sense and to explain their reasoning. Child 34's response was typical of the explanations that included spatial dimensions, which are terms describing the relationship between the dimensions of the parts and the whole: 'The pizza is cut into different parts—which can be big or small, but it's still the same amount of pizza altogether he is eating'.

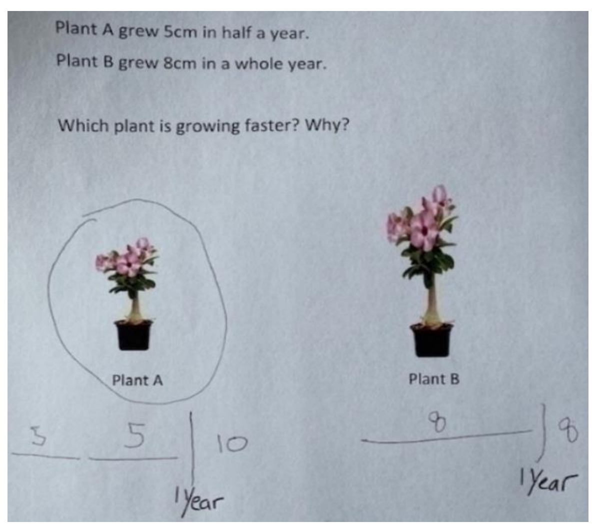
As described above in the intervention analysis, it is clear that many children have a developing understanding of partitioning, underpinned by the multiplicative inverse relationship. That is, an understanding of the number of parts and how these impacts on the size of each part. Similarly, the understanding of unitising and equivalence are evident in examples such as these; however, these items provided little or no stimulus other than the problem being read to the child. This required the children to draw on their mental imagery and understandings of the magnitude and quantities in each of the items. This suggests children developed this awareness of magnitude though their spatial reasoning skills in the absence of such representations.

Finally, Item 23 (Plant growth rates), which required children to compare the rate of growth for two plants, provides evidence of children's use of gesture and how this was integrated with their spatial structuring awareness. The item required children to consider the following problem: *Plant A had grown 5 cm in half a year, Plant B, that had grown 8 cm in a whole year. Which plant was growing faster?* Only four children were able to answer this problem correctly in the pre-intervention TBI, with little or no justification, suggesting they may have guessed. Of the 15 children who answered correctly in post-TBI, eight demonstrated a use of spatial structure, evidenced in part by the use of gesture, in the way they described and represented their answers (see Figure 5.38).

Figure 5.38

Children 38 (Left) and 43's (Right) Representations of Item 23: Plant Growth Rate





These two work samples are typical of the representations the children produced to explain their thinking. While the problem refers to plant growth rates in terms of height, the diagrams created by the children typically represented horizontal measures in the form of number lines to compare the rates of the two plants. Moreover, they recognised they were not representing the growth of the plants within the same timeframe; therefore, a representation that demonstrated doubling and halving structure was employed. Further analysis of this item revealed that eight of the 15 children who correctly answered this problem used gesture to represent the units of time and growth rates for comparison. This was in the form of gesturing the partitioning (or ‘chunking’ as many children spontaneously referred to it) of a year—either in the air with an imagined number line structure, or when explaining such representations as those in Figures 5.38 and 5.39. This finding suggests that the use of gesture was closely associated with how the children were visualising the proportional relationship between the two measures and early multiplicative relations of fractions and simple ratio during this TBI.

5.6 Chapter Summary

This chapter set out to provide insights into the way in which young children drew on spatial reasoning to develop an extended range of fraction ideas. With reference to the connection between spatial reasoning and early fraction understandings, the following insights were revealed in this iteration. Spatial visualisation assisted the children to conceptualise the relationship between the size of parts created is determined by the number of parts or shares. It helped children to visualise the process of partitioning and rearranging or redistributing parts of objects, which incorporated many early fraction as an operator ideas.

The children’s overt use of gesture in addition to the use of spatial language enabled the children to describe the size, orientation, transformation, or movement of the objects as the children were explaining their thinking, suggesting they were important tools and strategies for identifying how children were visualising and predicting various partitioning contexts. There were 116 coded instances of gesture recorded in the post-assessment (compared to a total of 36 instances in the pre-assessment). The analysis of gesture in the comparison of the pre- and post-TBI is important because it was not possible to view and record every child's use of gesture throughout the intervention itself due to the nature of the data recording tools. However, the post-TBI analysis suggests it was an important communication tool for the children and provides an insight into how children mentally represented and constructed these ideas.

The pattern, repetition and symmetry children paid attention to and engaged with when using concrete materials and pictorial representations was evident in both discrete and continuous contexts, suggesting spatial structuring was an important construct in developing an appreciation for the relationship between parts and whole. Specifically, spatial structuring was associated with fraction as a measure meanings about unit and composite fractions and how to coordinate these units in various ways across different fraction meanings. In addition, spatial structuring supported the fraction as a relation contexts involving discrete distribution and simple ratio activities.

Spatial proportional reasoning appeared to assist children in the justification of the size of the part and its relationship to the whole, particularly when identifying (visually) the halfway point within a nominated region (such as pathway or continuous object specifically). The children also developed proportional equivalence ideas from comparing fractions between different wholes. This suggest that the children's ability to compare and reason with different fraction measures was promoted through spatial proportional reasoning.

Reflecting on the classroom environment in which this teaching experiment took place, it was clear that the children were capable of developing understandings of a range of fraction ideas even though the pedagogical approach in the intervention program appeared to be strikingly different to the typical pedagogical approach the children experienced in their everyday mathematics lessons.

From a methodological perspective, the first iteration of the teaching experiment has satisfied what Cobb et al. (2001; 2003) describe as an analytical approach to DBR, in that it 1) has enabled the collective mathematical development of a classroom community to be documented through implementing an intervention-based instruction sequence, 2) enabled the documentation and analysis of the mathematical reasoning of individual children throughout the intervention and 3) resulted in analysis that feeds back into future iterations of the teaching experiment.

It was evident that the local instruction theory provided children with opportunities to develop an extended range of ideas from the fraction as measure, fraction as an operator, and fraction as a relation meaning, through spatial reasoning constructs. This is further supported by the results from the paired sample sign test, which revealed that Class B demonstrated statistically significant change from pre- to post-intervention TBI in all three meanings of fractions. While some of the tasks in the intervention appeared more challenging than others, the tasks still enabled every child to participate meaningfully throughout the intervention. The sequence and characteristics of each of the key indicators appeared to support the logical construction of fraction understandings throughout the intervention. This suggest there is no need to refine or change the local instruction theory at this point in the study.

Chapter Six reports the results of the second iteration of the teaching experiment for Phase Two.

Chapter 6: Teaching Experiment Insights – Class C

6.1 Chapter Overview

This chapter presents the findings from Class C as the second iteration of the teaching experiment. During this iteration, the children experienced only five of the 13 lessons within the intervention program. This was due to the COVID-19 pandemic and resultant restrictions on fieldwork by the University, and the South Australian Department for Education.

This chapter begins with an overview of the participating class in section 6.2. This section includes an analysis of the Task Based Interview (TBI) conducted prior to the intervention. Section 6.3 discusses the implications the pre-intervention TBI and classroom observations, and the modifications made to the intervention program consequently. Section 6.4 analyses the intervention findings in relation to the local instruction theory. As the intervention was cut short, only two key indicators of the local instruction theory are explicitly evidenced. To explore the impact of the intervention and the veracity of the local instruction theory, Section 6.5 presents quantitative and qualitative comparisons of the pre- and post-intervention TBI data. Section 6.6 summaries the key insights from this iteration of the teaching experiment.

6.2 Setting the Scene: Class C

The class participating in this iteration was from a third public school in regional South Australia. The Year 2 class comprised 21 children—11 boys and 10 girls. The mean age was 7 years, 2 months.

6.2.1 Understanding the Classroom Environment

As with the previous class, I undertook three classroom observations in the week before the intervention commenced. Using the CLASS framework (Pinta & Hamre, 2009), the observations were analysed to understand the typical ecology of the classroom and the needs, strengths, and behaviours of the children in their mathematics lessons. Teacher C used an interactive whiteboard during mathematics lessons to introduce the concepts or topics using PowerPoint, storybook, or online resources (e.g., songs, YouTube clips). In addition, the children were provided with concrete manipulatives such as multi-base arithmetic blocks, counters, dice, cards, board games, linking cubes and rulers, as well as a range of printed representations in the form of tens frames, number lines and place value charts as required.

The observations took place in early March 2020, approximately six weeks into Term 1 of the school year. While each of the schools participating in this study were demographically similar to each other (see Section 3.5), there were many children in this class that had complex home environments affecting children's attendance and school engagement. Some children had experienced trauma, and several had diagnosed learning difficulties (e.g., Dyslexia, Attention Deficit and Hyperactivity Disorder, Autism Spectrum Disorder, and global developmental delays). It was apparent in the observational period that the learning profile of this classroom (Class C) was quite different to that of the previous two classes (Class A and B).

Teacher C stated they had focused on linear measurement, subitising and two-digit place value in the lead up to this intervention, but focused on the Australian Curriculum Year 1 (version 8.4) content, as the majority of the children had not mastered many of these requirements. The three lessons observed were on ideas relating to place value. The first task was for children to play in pairs and take turns rolling two six-sided dice, adding the numbers rolled each time until someone reached 30. The teacher asked the class to represent the running totals using base ten blocks and a place value chart to understand the connection between the written numerals, their quantity and place value. As a new number was rolled, the children were asked to use the base ten blocks to add to the existing quantity and then record the total in numerals.

Teacher B modelled to the whole class how to ‘trade’ ten ones for one ten using base ten blocks. When the children began to work independently (in groups of three), Teacher C observed groups of children for a sustained time without interruption (approximately seven minutes per group), documenting their behaviours before identifying what support and scaffolding would be beneficial. In this lesson, Teacher C noticed that two children had difficulty representing the number 14. Teacher C asked the pair to consider the number they had written down and what each digit represented (one of the children had written 41 and required prompting by the teacher to establish whether they had recorded the intended number correctly).

Teacher C gathered several children on the floor whom they had also observed having difficulty making or recording different collections to provide some explicit teaching, which Teacher C described as a teachable moment. Teacher C asked the group of six children what each digit represented in 14. Most stated that the ‘1’ in 14 represented *one*, suggesting a reference to one as a single count (not one ten as a composite unit), and that the ‘4’ represented four ones. However, they became confused when this description did not match the number of cubes they had. Teacher C questioned them by asking them to line up the 14 individual units they had collected, then took a ‘ten’ multi-base arithmetic block and asked them if there was a way for them to use this unit in place of some of the ones. With some additional scaffolding, the children could eventually represent the number 14 as 1-ten and 4-ones.

The other two lessons I observed were similar, where Teacher C asked the children in groups to each roll two dice and make the highest or lowest two-digit numbers possible. The children then had to then represent the number using pop sticks (bundles of tens and individual sticks were provided), and then the children were asked to order the numbers. There were several classroom interruptions during the second and third lessons, meaning there was only approximately 30 minutes of working time in each observation.

Using the CLASS framework (Pinta & Hamre, 2009) to analyse the observations, the following themes were evident. First, Teacher C's experience working with diverse groups of children was evident in how they built a foundation for understanding by providing challenging but achievable tasks. This indicates that the *instructional support* and *classroom organisation* domains were evident in these lessons. For example, the *concept development, quality of feedback* and *language modelling* elements of the instructional support domain were obvious in how Teacher C encouraged the children to describe their thinking and reasoning. Teacher C often used phrases such as, 'Can you tell me what you were thinking when …?' or 'What is another way you can describe this …?', which helped the children connect the idea to the language the teacher was modelling and promoting. Teacher C would often paraphrase the children's responses to provide *targeted feedback* while offering an additional description to help the child connect their thinking (e.g., 'I like how you described 14 as 14 ones and one ten and four ones … can we use these ideas to describe 24?'). There appeared to be an emphasis on using the children's explanations to help them think and work through the problems.

There was no hurry to push children on to the next task; instead, Teacher C made adjustments that allowed the children to build confidence and competence with one idea at a time as needed (such as the example of representing 14 above). This indicated that the *emotional support domain* was a core component of the teacher–child interactions. For example, Teacher C appeared to *create a positive climate* by frequently using phrases like, 'I like the way you…' to start a conversation or direct the children's thinking in a particular direction. In addition, they would frequently ask a child (or small group of children) to explain their thinking to another child or group, showing sensitivity to different children's challenges and successes. When I asked Teacher C about this approach, they stated that the children needed much repetition to develop ideas (in all learning areas generally). However, they felt this strategy helped achieve this for mathematics while building their confidence.

These observations provided insights for me as the teacher–researcher for this iteration, including the need to provide multiple scaffolds (such as the questioning and prompting techniques) and opportunities for children to work with each other.

6.2.2 Pre-Intervention Task-Based Interview Assessment Insights

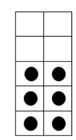
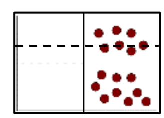
As described in Chapter Three, the pre-assessment TBI was conducted one on one and comprised of 24 items, divided into three sets. The first nine items were whole number based to assess two big ideas in number: Trusting the Count (Set One) and Place Value (Set Two) as part of the Assessment for Common Misunderstanding (AfCM) tools (Siemon, 2006). The remaining 15 items (Set Three) targeted children's fraction understanding and spatial skills. Set three items were created or adapted for this study based on the literature concerning both fraction and spatial reasoning research. The children's responses were scored as either correct, partially correct or incorrect as per the rubrics described in Chapter Three. The raw scores from the pre-assessment TBI are presented in Appendix H.

6.2.2.1 Set One: Trusting the Count Insights

The assessment items for Set One are summarised in Table 6.1

Table 6.1

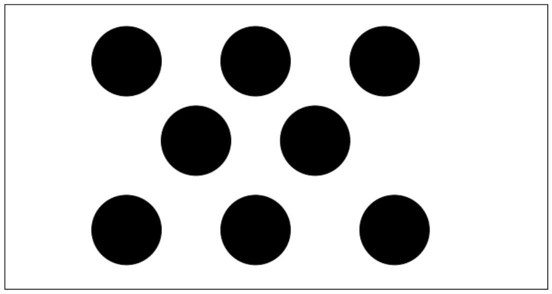
Set One Task-Based Interview Assessment Item Descriptions

Set One Assessment Items
Item 1: Subitising cards Cards 1–6 (common dot die arrangement) Cards 7–10 (tens frames and structurally ordered arrangements, e.g., triangular arrangement of dots for 10) Cards 7–19 (tens frames ordered and random)
Item 2: Hidden counters task Place five counters and bag in front of child, rattle to demonstrate that there are counters in the bag. Place four counters in front of child. *'There are four counters here and five more in this bag. How many counters altogether? How did you work that out?'*
Item 3: Tens frame bananas Children are asked to think about the dots on the tens frames as bananas. *'If I have this many bananas, and three more bananas were added, how many are there altogether?'* 
Item 4: Hidden Dots task *'There are seven dots here (in the top section) and nine dots here (bottom section)'.*  Cover the '9' card with the flap and ask: *'How many dots altogether? How did you work that out?'*

The 21 children in Class C predominantly achieved an average of partially correct results in Set One of the assessments. For Item 1: Subitising cards, 12 children could subitise to 10, but typically only when the dots were presented in an ordered or familiar pattern—such as dot die arrangements or tens frames. The collection of eight was presented to the children as shown in Figure 6.1.

Figure 6.1

Subitising Card for Eight

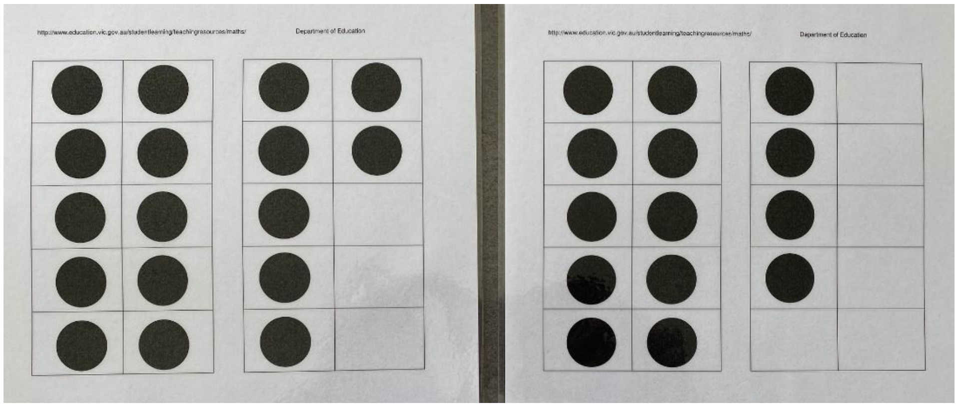


Six children said 'eight' without counting. When invited to explain, the children referred to seeing 'the lines' of dots; that is, either as two lines of three and two more, or, that the top and bottom lines were the same and the middle was one less. Alternatively, one child stated they recognised eight as five and three more. This child (Child 49) also stated that the three dots beside the familiar star-like arrangement of five was 'like its shadow…because it's the same', referring to the shape and arrangement of the collection. In both types of descriptions (three, three and two; or five and three), the children's responses indicated they are all paying attention to the structural arrangements and repetition within the patterns to determine the quantity.

Two children were able to conceptually subitise all collections up to 19. The following response from Child 56 provides some interesting insights into their thinking (see Figure 6.2).

Figure 6.2

Subitising Tens Frame Cards: 17 and 14



Child 56 explained:

> For the 17 card, you can see the first two rows are a ten. But I looked at it like there are three rows of five, because one row is five, and so three is 15 and two more is 17. In the 14 card, I went backwards: there’s three rows of five but one missing, so it’s 14.

Child 56’s explanation suggests they have a mental object for five and used this unit to identify larger collections such as 17 and 14. Their response indicates they were using the structure of the tens frame to recognise countable units of five. That is, the child is using the unit of five in flexible ways such as three fives (15), two more than three fives (17) and one less than three fives (14). This suggests they have an advanced structural awareness of the quantities presented in this task, because they were demonstrating a generalised understanding of the collections.

While the examples of Children 49 and 56’s thinking demonstrated flexible ideas about whole numbers, most of the children in Class C demonstrated they could only subitise (most) numbers to 10 when the arrangements were presented in familiar arrangements (such as dot die arrangements and tens frames). However, this insight suggested that spatial structure plays an important role in the children’s ability to recognise and work with such quantities. As described in the previous teaching experiment, spatial structuring is an awareness of the pattern and arrangement of objects that can assist in determining mathematical relationships such as part-whole relations of quantities. The children's awareness of spatial structures in this activity suggests children may also draw on these initial understandings to explore fraction ideas in discrete sets within the forthcoming intervention.

The second insight from Set One was children's inability to work with hidden collections. For example, in Item Two (Hidden Counters Task), 10 children were observed counting on their fingers in place of the five counters hidden in the calico bag, or they would tap on the bag five times and then used one-to-one correspondence on the visible four counters to enumerate the collection. This emergent part-part-whole understanding was also evident in Item Three (Tens Frame Bananas). Nine children indicated partial success by drawing or placing counters on the tens frame of six (many often recounting the six dots to begin with). According to Steffe (2001), this behaviour suggests the children are figural counters in that the child uses visual or auditory cue to help keep track of the count. That means children can generally only count what they can see and have trouble with counting hidden collections (or parts of the collections). This suggested that children will need time to develop mental models of different quantities to interiorise the quantity and establish the relationship between groups as composite units and the total quantity (Steffe, 2001). This insight is consistent with Teacher C's comment in the previous section that children need lots of repetition to develop an idea.

6.2.2.2 Set Two: Place Value Insights

Set Two was designed to determine some of the key underpinnings of place value, such as name, compare and rename collections in terms of their place value parts. Table 6.2 presents a brief description of each task.

Table 6.2

Set Two Task-Based Interview Assessment Item Descriptions

Set Two Assessment Items
Item 5: Counting 26 counters Child counts collection and records. Circle the ‘6’ in ‘26’ and ask, *‘Does this have anything to do with how many counters you have there?’* Circle the ‘2’ in ‘26’ and repeat the item. Ask child to explain their thinking if not obvious.
Item 6: Place-Value Bundles 13 bundles of ten pop sticks and 16 single sticks are provided. Child is asked to make 34 using these materials.
Item 7: More than/Less than… Card with 86 is presented to the child. ‘*Write the number that is one more than this number? Write the number that is one ten more than this number?’* If correct, say, *‘Write the number that is three less than this number? Write the number that is two tens more than this number?*’ Ask child to explain their thinking if not obvious.
Item 8: Proportional number line task Place the 0 to 20 Open Number Line Card in front of the child and say, ‘*Use the pencil to make a mark to show where you think the number 8 would be. Why did you put it there?’* Repeat with the number 16. If reasonably accurate and/or explanation plausible, turn the card over to show the 0 to 100 open number line and say, *‘Make a mark to show where you think 48 would be. Why did you put it there?’* Repeat with the numbers 67 and 26. Ask child to explain their thinking if not obvious.

The modal response across the five items in this set was ‘No/ Incorrect response’. The children typically demonstrated little understanding of how two-digit numbers are constructed. Using bundles of 10 was challenging, as was identifying one more, three less, or one ten more than or less than a given two-digit number for over half the class. This behaviour suggested the children have not necessarily consolidated ideas about viewing 10 as a countable unit. For example, some children could count by tens using the multiple of tens counting sequence (e.g., 10, 20, 30…) but could not represent 34 with bundles and ones, suggesting they may have been memorising skip counting by 10 rather than understanding place value unit ideas concerning the materials provided. While some children demonstrated an additive understanding of place value in terms of stating 26 and 34 were a 20 and six, 30 and four, respectively, they were unable to consistently work with a count of tens and a count of ones independently, as evidenced by their inability to count by twos with the counters and single pop sticks or using 10 (bundles) as a countable unit (Rogers, 2012).

Twelve children had partial success with Item 8 (Proportional number line task), meaning they were able to complete part of the task only. Many children took a counting approach by placing their finger at 0 and counting imagined partitions on the 0–20 number line to place eight and 16; however, most children attempted to adjust where they placed the number eight if it did not seem proportional. This suggested the children were paying attention to the relationship between the size of the numbers and where they were placed in relation to each other on the number line. Yet, they typically gave no clear rationale for this in their reasoning. It appeared there was some intuition as to where they were placing the numbers, but little explanation was given, and often the children would try and revert to counting to verify where they placed each number. Only one child could complete the 0–100 number line. The typical responses from the remaining 12 children to the second part of this task were, ‘there’s too much counting’ (Child 61) or ‘I don’t know how to fit them [the numbers] in’ (Child 69). This suggested that the numbers are not well understood in terms of their relative magnitude (i.e., how ‘far’ eight is from 16; how ‘far’ 16 is from 20, etc.).

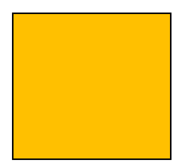
6.2.2.3 Set Three: Fractions and Spatial Reasoning Insights

Set Three was designed to assess children's fraction understanding and spatial reasoning capabilities. Some items included both fraction and spatial constructs, while others were designed to independently evaluate either a spatial reasoning capability or a fraction idea. The full description of the items in Set Three is presented in Chapter Three.

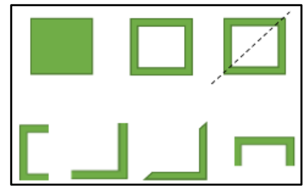
The pre-intervention TBI revealed that the children again predominately scored incorrect responses or could not provide a response for most items in this set. There were, however, four tasks from this set where more than half of the class scored a partially correct or correct response (see Appendix H). Interestingly, these were the same four items the children in the previous class (Class B) had the greatest success within the pre-intervention TBI. These were Items 10 (Folding Fractions), 16 (Halving the Stars), 18 (Gisele's paper square) and 19 (Scale the picture). Table 6.3 describes Items 10, 16, 18 and 19 for this analysis.

Table 6.3

Set Three Items 10, 16, 18 and 19 Descriptions

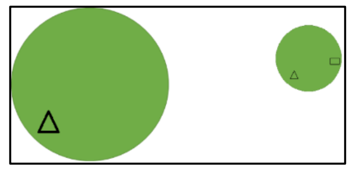
Selection of Set Three Assessment Items	
Item 10: Folding fractions Child is shown the image of a square. *'How many ways can you imagine folding a square in half? Can you describe what you think it would look like if you folded it in half, then in half again? What is each part called?'*	
Item 16: Halving the stars Child is presented with the image. '*If you gave away half of this collection of stars (16) how many would you have left?*'	

Item 18: Giselle's paper square



A series of folds is made to a square, and the child needs to identify what the end result would be from four possible options. The children were only shown the four possible outcomes to choose from.
'Gisele had a green sheet of paper and cut a white shape out of the middle of the paper. Then she folds the paper in half, diagonally. Which of the four shapes below did Gisele see?'

Item 19: Scale the picture



An image of two circles is presented. The diameter of the smaller circle is half that of the larger circle. The smaller circle contains two shapes, a triangle and a rectangle. The larger circle only includes the triangle drawn to scale. The children are required to draw the missing rectangle in the larger circle and describe the difference in size between the two shapes.

'Can you complete the picture of the circle on the left so that it has the same shapes as the circle on the right? Explain why you chose to draw the shapes in that way.'

Items 10 and 18 both focused on spatial visualisation and partitioning half an object—in both cases, images of different pieces of paper. In Item 10 (Folding fractions), two children noted several ways the square could be halved, stating the need for the parts to be the same size rather than the shape or arrangement of parts. For example, Child 65 drew vertical, horizontal, and diagonal lines across the square to demonstrate how they imagined partitioning in half. They then stated you could 'break these parts up though, and put them in two piles, as long as they are the same size… they're half'. Sixteen children described how they visualised or drew at least one way a square could be partitioned in half. For example, seven children referred to creating halves by folding in a cross formation (where these children used the term 'cross' with an associated gesture to indicate a diagonal and/or perpendicular fold, typically using a finger in the air to illustrate).

Further, 11 children referred to geometric properties in terms of the spatial dimensions of the unit fraction when describing how halves could be made from the square. An example is from Child 69, who stated, ‘you can make a triangle half or a square half [referring to a rectangle]’. This is an example of the partitioning concept. That is, the children suggest they understand that half is an equal part in relation to its relative whole, rather than using half as a synonym for creating two parts of any size. However, not all the 11 children provided such a sophisticated response.

In Item 18 (Giselle’s Paper Square), 12 children indicated they were paying attention to the geometric symmetries and structure in the way they identified the correct answer as 1-half of the pre-folded paper. For example, Child 55 stated, ‘you can tell [which is the correct shape] by the way the corners have been cut’, referring to the diagonal partition down the middle of the paper square. Child 61 stated, ‘if you imagine folding it [diagonally], the half and half would look like this. They are the same but facing each other’. This statement suggested the child visualised the folding of the square in a diagonal form and how this contributed to the structure of the halves as triangular parts. Further, these descriptions demonstrate the connection between geometric symmetries—an idea within Confrey et al.’s. (2014b) rational number learning trajectory framework explained in Chapter Three.

In the previous two items, 15 children used gestures while thinking about the problem and formulating a response. For example, when describing how to fold the square piece of paper in Item 10 and how they would physically fold or cut a paper square in Item 18. For example, Figures 6.3 and 6.4 indicate how children gestured folding and cutting for partitioning the square.

Figure 6.3

The 'Folding' Gesture

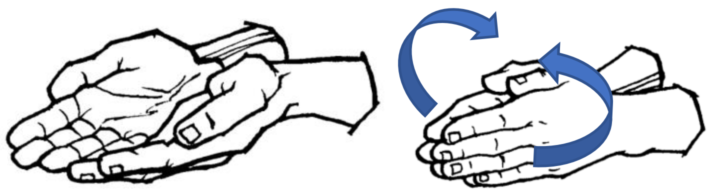


Figure 6.4

The Cutting Gesture

In Figure 6.4, the left hand in this gesture typically moves forward and back over the right hand, signifying a sawing or cutting action.

The following two responses indicate how the children may have engaged in spatial visualisation during this task and how their use of gesture assisted in this explanation:

> Child 62: I know if I folded in half this way [used the folding gesture shown in Figure 6.3], and then folded again [used a similar gesture indicating they imagined the paper folded into four vertically positioned parts] would give you four parts the same.
>
> Child 57: I just thought…you can cut halves like this [gestured the cutting action shown in Figure 6.4 to produce a horizontal, vertical and one diagonal partition across the square].

The use of gesture continued in Item 16, with which the children had the greatest success. Item 16 (Halving the stars) required children to determine how many stars would be left in a set if half were given away. This item was completed by 13 children, who appeared to subitise two groups of four stars on either side of an imaginary line they said they visualised, indicated in Figure 6.5.

Figure 6.5

Item 16 Stimulus: If We Gave Away Half of These Stars, How Many Would We Have Left?

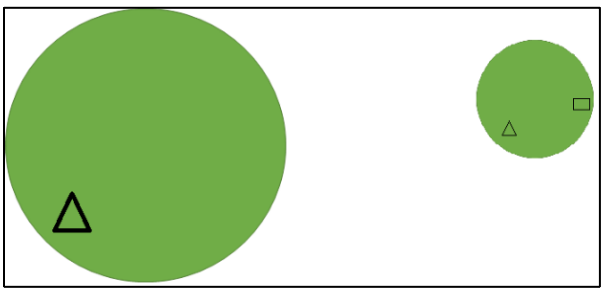
Note. Arrow indicates the position of the line the children indicated they visualised.

All 13 children either ran their fingers down the middle of the collection or placed their hands in the position indicated in Figure 6.5 to suggest how they visually partitioned the stars. Three children also stated ‘four and four’ with fluency, indicating they subitised rather than counted each smaller collection. Although the children successfully recognised the two groups of four stars were ‘matching’ and ‘equal’ in terms of quantity, 10 of the 13 children that successfully answered this item still counted each star on both sides of the imagined partition before stating ‘eight’ is half of the collection. When asked about their counting, they typically described perceiving the set of stars in two symmetrical parts. However, they reverted to counting (observed by head nodding or pointing to each star) to enumerate the half. Here, their part-part-whole knowledge is emergent and seems to be driven by the image’s geometric symmetry (Confrey et al., 2014b). This item intended to explore the fraction as an operator idea of 1-nth-of... and halving; however, it can also be solved with whole number thinking and counting strategies, evident in the children’s responses. This is a weakness of the TBI that will be discussed in Chapter Eight; however, it does provide further evidence for how children may develop the fraction ideas in parallel to their whole number knowledge.

Item 19 (Scale the picture) incorporated doubling/halving ideas in addition to spatial proportional reasoning. The task required children to consider the two images of a shape that had been enlarged. The larger circle had a diameter double that of the smaller circle. The edges of the triangle had doubled, and the children were asked to draw the missing rectangle on the larger circle to scale (see Figure 6.6).

Figure 6.6

Item 19: Scale the Picture Stimulus



In addition, the children were asked to describe a plausible relationship between the two images. It was not expected that the children would understand or recognise the increase in diameter measure per se. Instead, the intention was for them to notice and describe the proportional relationship between the two images (e.g., the circle on the left is a larger version of the circle on the right; the triangle's edges on the left are twice as big as the triangle on the right, etc.).

While 17 children were able to draw the missing rectangle reasonable accurately to scale on the provided image, most could not articulate a relationship between the two images, other than the first circle was larger than the second (no mention of the proportional relationship between the triangle or rectangle's size). Most children who scored partially correct needed prompting about describing why they chose to draw their rectangle in the position and size they did, with many stating the rectangle they added ‘just looked right’. Even so, many children who completed this item were observed running their fingers around the boundary of the two circles or using their forefinger and thumb in what appeared to be an informal measuring gesture to compare the two objects. The gesture appeared to be a strategy the children used to communicate their emergent spatial proportional reasoning awareness.

6.2.2.4 Summary of Pre-Intervention TBI Insights

The pre-assessment TBI analysis provides several key insights that help describe the children’s baseline understandings of whole number and fraction ideas and their spatial reasoning capabilities. First, the results of Set One and Two of the TBI assessment suggested the children had limited understandings of part-part-whole understandings based on their performance on the subitising tasks, where typically the children could only subitise to 10 and if the dots were in structured or familiar arrangements.

Set Two revealed the majority of the class had difficulties with place value ideas and demonstrated they have not established the idea of 10 as a countable unit consistently. Although some children indicated some intuition to where numbers are placed on a 0–20, there was often a counting-based approach taken and little explanation given, also indicating that one- and two-digit numbers are not well understood in terms of their relative magnitude within this assessment.

The results revealed that the children were similar in their experiences and understanding of fraction ideas and spatial reasoning abilities to the previous class, as they demonstrated similar levels of success on the same four items (Items, 10, 16, 18 and 19). Some children were able to create fair shares or the unit fraction of half in some contexts with attention paid to the geometric aspects of the representations, meaning there was some fraction as an operator and fraction as a measure understanding. A common thread linked these four items analysed in Set Three, suggesting gestures were critical elements in children's conceptualisation of the problems. This inference is made because of the high frequency of gestures observed when the children problem-solve or justify their answers.

Finally, despite the children in Class C performing to a similar standard to Class B on the TBI, there was a notable difference in how long the TBIs took for Class C, which was related to the observation analysis. Typically, each assessment took 30 minutes or more because the children needed more time to think and process the questions, or they became distracted from the task and required a break in the middle of the interview.

6.3 Implications for the Teaching Experiment

Based on the analysis of the classroom context through the observations and pre-intervention assessment data, I was mindful that the children may need adjustments throughout the intervention. I anticipated that this class may need more time to process and explore unfamiliar topics than was required in Class B. It was also a challenging time emotionally for the children and teacher involved due to the unfolding COVID-19 pandemic, so I wanted to ensure the intervention did not cause any additional undue stress or anxiety due to the unfamiliar content.

Table 6.4 summarises the changes to the intervention program as it was implemented, including the eventual disruption caused by the COVID-19 pandemic.

Table 6.4

List of Changes Made to Each Lesson as It was Implemented

Lesson	Original activities planned	Changes made for Class C
Lesson 1: Sharing cookies	Introduction to *The Doorbell Rang* (Hutchins, 1979) and fair shares of 12 cookies.	No change.

Lesson 2: What is a fair share?	Exploring fair and unfair shares in discrete and continuous contexts.	No change.
Lesson 3: Visualising the share of a cookie	Exploring quarters of different geometric shapes. Visualising the share of a cookie from repeated halving, thirds, and sixths.	Visualising the share of a cookie to explore partitioning one cookie in halves and quarters only (see explanation below).
Lesson 4: Sharing easily divisible collections	Exploring shares of 10 cookies, Three shares/3-fifths/three times as many-as-one share.	Replaced by Lesson 7: Tablecloths (see explanation below).
Lesson 5: Cookie fraction estimation	Visualising the size of parts missing from a cookie. Visualising composite units of cookies created from unit fractions (e.g., 1-quarter, and 1-half = ¾ of a cookie)	Children were introduced to the focus of the original Lesson 4: Sharing easily divisible collections.
Lessons 6–13	Abandoned due to COVID-19 restrictions.	

For Lesson 3: Visualising the share of a cookie, the activities were originally planned for children to explore the outcome of partitioning one cookie and other geometric shapes by repeated halving, and then progress to sharing between three and six friends. As the children appeared to have little confidence with partitioning, this lesson primarily focused on visualising and then comparing the outcome from repeated halving of a single cookie and other geometric shapes (i.e., rectangular paper strips) rather than thirds and sixths.

For Lessons 4 and 5, changes were made to the sequence of tasks. In the debrief with Teacher C after the three pre-intervention observations, we noted that many children appeared to have difficulty holding a single idea or remembering an idea, such as creating equal parts in both discrete and continuous context simultaneously. While the children would still engage in the first two lessons as planned, I decided to introduce Lesson 7 (Tablecloths) on day three to help promote the understating of fraction magnitude and how equal parts are created. This lesson continued to focus on visualising and comparing equal and composite parts in continuous contexts and still promoted the second key indicator of reinitialising the unit. The reason for this was to help support the children to continue constructing their understanding of partitioning by having a more sustained focus on continuous contexts than was previously planned. The children were then introduced to the focus of discrete sets from the original Lesson 4 activity (Sharing easily divisible collections) after they had completed Lesson 7 (Tablecloths) to help them build fluency in a fair share between both continuous and discrete models and their awareness of creating and naming different shares within a given whole—the first two key indicators of the local instruction theory. However, not all children were introduced to the discrete contexts due to difficulties experienced with competing the prior tasks.

6.4 The Teaching Experiment: Insights from the Intervention Program

This section presents an analysis of the lessons experienced by the children in this iteration of the teaching experiment. As previously stated, Class C only participated in five of the 13 intended lessons due to the COVID-19 pandemic. Based on the rapid introduction of restrictions and government stay-at-home orders, I made the decision to stop the intervention on day five and commence post-intervention TBIs so that I had assessment data for comparison. Two days after I finished the TBI assessments, the South Australian Department for Education excluded all non-essential persons from entering school premises (including parents), which lasted for the duration of 2020. In addition to these restrictions, RMIT University suspended all research fieldwork (a restriction that remained in place for much of the next year and a half).

Like Chapter Five, the structure of this discussion of the relationship between the spatial reasoning constructs and the children's development of the intended fraction ideas is based on the key indicators of the local instruction theory, represented in Table 6.5.

Table 6.5

The Local Instruction Theory (Version Three)

Key Indicators	**Characteristics of Tasks**	
	Primary Fraction Foci	**Spatial Reasoning Approach**
Creating and justifying equal shares	**Fraction as Operator:** Fair shares Doubling/ halving Partitive division/ recursive multiplication, Geometric symmetries, Similarity **Fraction as Measure:** Many-as-one, Measure, Composite units, Unit fraction	Visual perception of equal groups (drawing on spatial structures and arrangements). Equality of parts regardless of model (i.e., equal parts for discrete collections and continuous models less than and greater than 1). Visual awareness of the geometric properties of parts and sets (e.g., shape, orientation, pattern, symmetry). Observing the physical transformations of partitioning (dividing and reassembling), and visualising and predicting the outcome of a nominated split (e.g., spatial visualisation)
Reinitialising the unit	**Fraction as Measure:** Composite units, Unit fractions, Part-Whole fractions, Equivalent fractions **Fraction as Operator:** Fair shares, Doubling / halving; Partitive division/ recursive multiplication, Times-as-many, Similarity **Fraction as a relation** Many-to-one	Visualising measures between parts and wholes, and between composite and unit fractions through unitising. Exploring the spatial structure and arrangement of objects and sets to create and compare different units fractions. Visualising magnitude relations between parts (double/ half/ times as many) the distribution of parts to determine equivalence.

	Distribution	
Recognising proportional equivalence	**Fraction as a relation** Distribution, Proto-ratio, Equipartitioning multiple wholes, **Fraction as Operator** Doubling /halving Times-as-many, 1-nth-of…, Scaling, Geometric symmetries, Similarity **Fraction as a measure** Composite units Unit fractions Equivalent fractions	Visualising the relationship between equivalent measures, of same and different wholes.
Connecting multiplicative relations	**Fraction as Relation** Many-to-one, Distribution, Proto-ratio **Fraction as Operator** Doubling/ halving Partitive division/ recursive multiplication Times-as-many, 1-nth-of.., Scaling **Fraction as Measure** Composite units Part whole fractions Equivalent fractions	Early relational understandings between the structure of part-part and part-whole quantities. (e.g., visualising and justifying the relational magnitude of fractions in relation to other fractions (e.g.- quarter is a half of a half/ twice as small); and working flexibly with non-symbolic simple ratios (e.g., 1:2 = 2:4).

Due to the reduced timeframe of this teaching experiment, not all key indicators were explicitly explored. This chapter will now present evidence of the following two key indicators: *creating and justifying equal shares* and *reinitialising the unit.*

6.4.1 Key Indicator: Creating and Justifying Equal Shares

As detailed in Chapter Four, the first key indicator of the local instruction theory was creating and justifying equal shares in both discrete and continuous models.

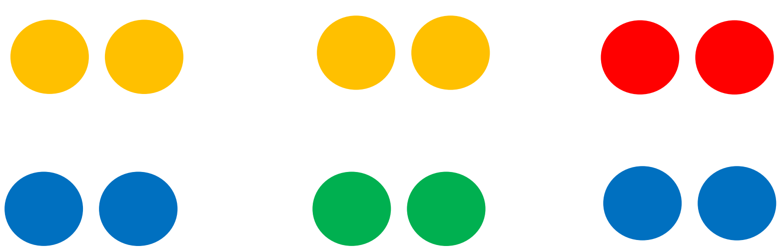
6.4.1.1 Lesson 1: Sharing Cookies

In Lesson 1, the children were asked to share 12 cookies between two, four, six and eight children and identify the relationship between the shares created and the size of the shares. They were provided with counters, paper circles and a task sheet to record their thinking. I spent some time asking children how they might use each of these materials with the context of six cookies, as it appeared they needed much prompting and scaffolding to start. I also asked the class to pause periodically throughout this activity and asked different children to share their thinking and strategies for the sharing situation they were working on at the time. This was consistent with how Teacher C used examples of children's thinking during the working time to help promote other children's understandings and strategy choices.

Child 65, who I observed using the counters available to work through the problem of sharing 12 cookies between six people (see Figure 6.7), provided the following insights.

Figure 6.7

Recreation of Child 65's Representation of 12 Cookies Shared Between Six People



I asked Child 65 to tell me about what they had done and what their representation meant. They replied:

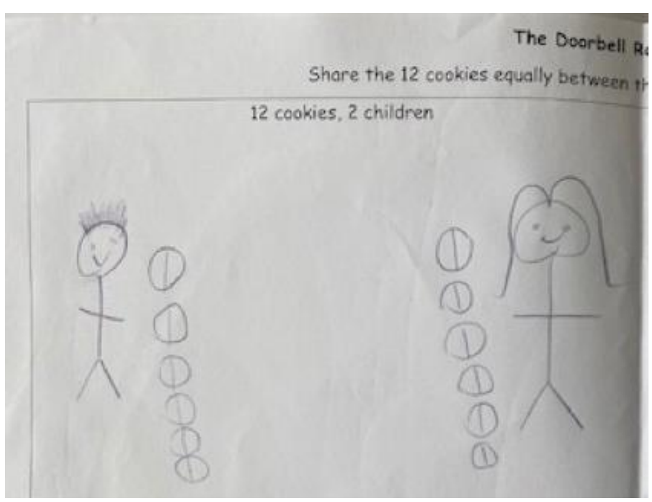
> There needed to be six groups, because there was six people, so I started to put the counters out into six [groups]. But I lined them up so you can see them in twos [moving their hand in a horizonal action back and forth]. This way, you can see two, two and two [pointing to the top row of six counters], and it's the same [on the bottom row] two, two and two, so you know it makes 12—but you've just split them into groups…six groups of two cookies.

Child 65's systematic way of describing how they viewed or considered the arrangement of the quantity indicated the partitive division idea to create the shares, but they were unable to conceptualise what the shares meant as a fraction of the set at this point; rather, they were exploring this set as a whole number context through multiplicative units. This representation revealed that spatial structure was an important element to the child's explanation and thinking.

Similarly, children's drawings revealed the connection between fair sharing and partitive division ideas of the fraction as an operator meaning. Child 62 used gesture to describe their thinking when representing the following partitions on the story board (see Figure 6.8).

Figure 6.8

Child 62's Work Sample



Child 62 chose not to use any concrete materials for this problem. In this work sample, I was particularly interested in the representation of 12 cookies between two children, where this child had drawn two columns of six cookies, and each cookie also appeared to be partitioned in half. I asked Child 62 to explain their drawing. They replied:

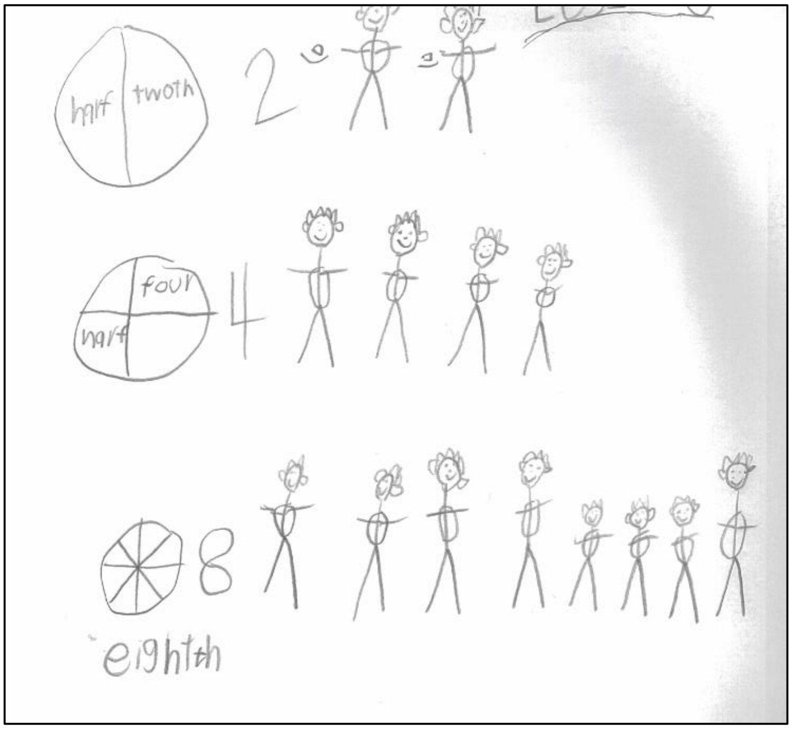
> When I drew the two different groups, I knew each kid would get six—a half of 12 is six. But then I just cut each cookie in half too, because you can see [using hand perpendicular to the page]…12-halves here for this person, and same for this one—two rows of halves [using hand to gesture linear groups of partitioned cookies]. (Child 62)

Child 62's responses could indicate a sophistication in their number knowledge, as they demonstrated a flexibility between whole number and fractional parts in their understanding of 12, indicating reinitialising the unit. However, their representation could also indicate a confusion between half of the collection and half of each item within the collection. The former is implied, as Child 62 named 'how much' each group is worth in each share (i.e., one share is two 'rows' [columns] of 6-halves), indicating both partitive division (sharing a quantity between a given number) and the many-as-one idea from the fraction as a measure meaning. Further, this child's use of gesture (in the form of indicating the arrangement of the shares they were discussing in their drawing) suggested they were using spatial structure to arrange the 12 cookies into two equal groups of 6-halves. The child's representation was not part of the initial problem, so it was a surprising representation for this child to draw; however, they had been sitting next to another small group of children who were working through the 12 cookies shared between eight children problem, where they were exploring how to create mixed fractions as a fair share, which may have prompted them to represent their thinking in this way.

Child 56 interpreted the task differently, in that they chose to represent sharing one cookie between two, four and eight people. Although this was different to what was asked, their representation and discussion that followed revealed insights into their understanding of equal shares (see Figure 6.9).

Figure 6.9

Child 56's Work Sample for Lesson 1



Child 56: I did it wrong, but it's still right [referring to sharing one cookie instead of 12]. Everyone needs the same amount, but you get more…when it's less people.

I asked the child to tell me more about their representation, including how they had labelled some of the parts.

Child 56: It's doubled [pause]…No, it's actually, halves. It goes from half to fourths to eighths. See, half of a half is fourths, half all the quarters you get eighths [pointing from cookie to the next].

I prompted the child further, 'Okay. If its halves, like you say, what would come next?'

Child 56: Ummm, that's double eight… but half of each eighth [running fingers over the cookie partitioned in eighths]…. ahh… [pause]… sixteenths!

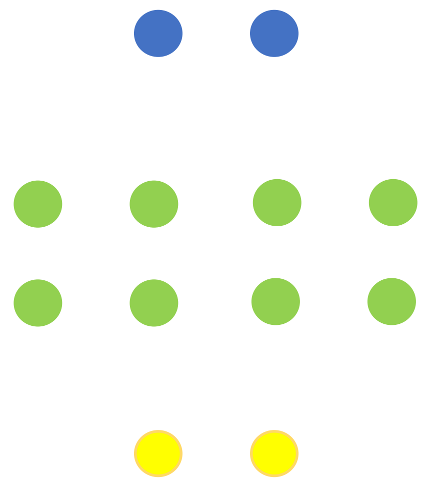
This description indicated that the child was conceptualising the doubling/halving and 1-nth-of... ideas within the fraction as an operator meaning. The child's description suggested that they knew the shares are equal because of the relationship between the number of parts and the size of the parts created though halving. Moreover, this child also revealed the ability to reinitialise the unit by naming units within units—the second key indicator in the local instruction theory.

The problem of sharing eight cookies between 12 people revealed some further insights into the relationship between the fraction as an operator and fraction as a measure meaning, and how spatial reasoning assisted in the development of the key indicator—creating and justifying equal shares. For many children, the context of sharing 12 cookies between eight children was not attempted in the first instance. Class C clearly struggled as a whole when they came to this task, so I took the opportunity to provide some intentional teaching after we had a short break. In the whole class discussion, I challenged the children to articulate what exactly the problem was asking, and after some discussion and scaffolding with concrete materials, eventually the class recognised that there were more cookies than people to share between, therefore everyone is going to receive one cookie and some more. While all children had access to concrete materials (counters and paper circles), I observed four children spontaneously gesturing a cutting motion with hands placed perpendicular to each other, similar to the cutting/sawing gesture in Figure 6.4.

The 'cutting/sawing' gesture suggested it was a catalyst for several other children recognising that a cookie could be partitioned to work out individual shares and that they could split the cookies to share out to the 12 people in problem. From this action, three children worked together to make the following representation using the counters (see Figure 6.10) and presented it to the class.

Figure 6.10

Arrangement of Representation from Children 52, 55 and 57

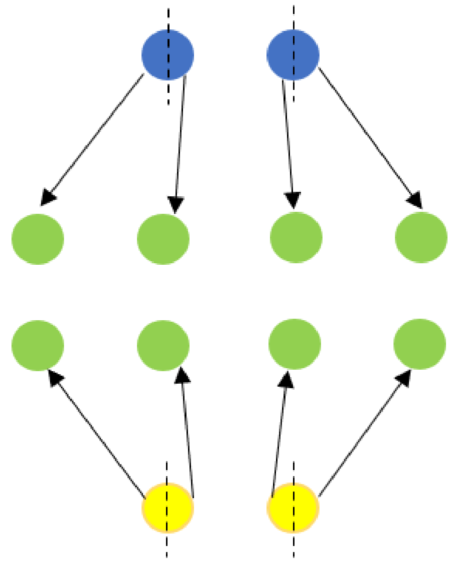


Note. The colour of the counters above were selected in this thesis to assist in the discussion.

> Child 57: We knew that there would be one cookie for each person, so we decided to line them up like this so you can see the four and four rows [gesturing horizontal movements with hand to indicate the central two rows of four]. This left four over, but to work it out it’s easy because these cookies [referring the two counters at the top and bottom of Figure 6.10] have to be shared between four people [see Figure 6.11].

Figure 6.11

Interpretation of the Visualised Process of Sharing



During this whole class presentation by Child 57, I could see some children watching on looking puzzled, so I repeated and rephrased what this child had described to the whole class, so everyone had the opportunity to understand the representation this far. Child 52 continued the group's explanation:

> This one cookie [referring to one of the two cookies represented in yellow in Figures 6.10 and 6.11] gets broken in half and shared to these two people [pulls hands apart suggesting they are dealing out the two parts of the yellow cookie and placing the parts on top of two green cookies] and same with this one [gestures the same partition for the two blue cookies].

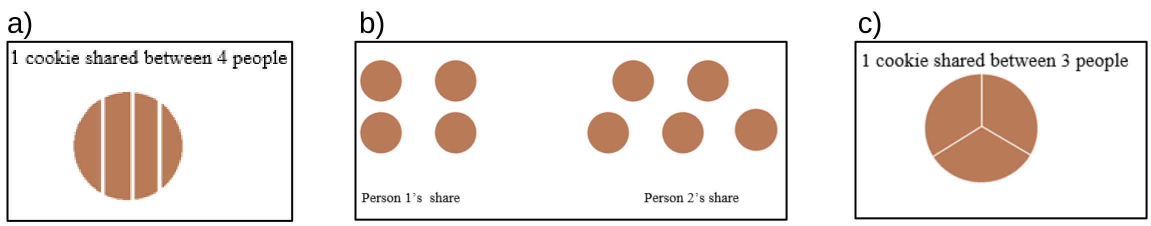
This explanation highlights how the children used spatial visualisation in the context of partitive division to imagine partitioning the remaining four cookies in half to create equal shares of the collection. This also illustrates the many composite unit and part-whole ideas from the fraction as a measure meaning and using gesture to help communicate their thinking of each measure, of the representation. Similarly, the spatial structure and arrangement of the cookies, whether in set or continuous contexts, appeared to be a critical element visually organising the shares.

6.4.1.2 Lesson 2: Creating Fair Shares

During the *Explore* phase of this lesson, the children were placed into small groups to explore images of cookies that had been shared both equally and unequally (see Figure 6.12). The cards were laminated so the children could not physically manipulate the parts, and the children were asked to justify how they knew the shares were fair or unfair.

Figure 6.12

Examples of Cookies Representing Fair and Unfair Shares



Although fair shares is an idea derived from the fraction as an operator meaning, the intent of this task was to also consider the individual parts as a measure (i.e., are the parts fair [operator] and how much of the whole do the parts represent [measure]?). Further, the intent of using the cards was to provide an opportunity for children to draw on their spatial visualisation skills and awareness of the spatial structures of the objects, sets and parts to determine their answer.

Two groups of three children were observed playfully stacking their hands on top of each other (similar to the card game of 'snap'). I asked them to explain what they were doing, and Child 67 replied:

> Me and [Child 63] thought that in the one that had four parts [referring to one cookie shared between four people]…it's lining them up, on top of each other [child used their hands to represent a stacking gesture]…you see these [points to middle two parts of the cookie image] are big ones.

Their explanation suggested they were using their hand as an informal measuring tool, to describe that the middle two parts of the cookie were bigger than the two edge pieces. Child 70 added:

> If we used our hands like the parts of the cookie they're all uneven… the edge parts are smaller [than the middle two parts] … but joining the two parts on each side does make an even, same size half [gesturing semi-circle figure with their hands].

I asked the group if it were possible to share a cookie between four people equally, given they had determined the current image had unequal parts. Child 69 stated, 'yes…cutting across it will give you fair shares—because the parts are the same'. With this explanation, the child used their hand to gesture a horizontal and vertical cut across the centre of the cookie picture.

These responses and interactions suggested that the children were paying attention to how mentally moving (visualising) the parts on top of each to compare the different measures, represented by their use of gesture, in this case, helped determine that the parts were unequal. In addition, they indicated a many-as-one idea by combining the two unequal parts on either side of the mid-line to visualise and recognise the halves. Although the two parts used to create 1-half were not equal to each other, the child recognised that the geometric structure of the two parts, when combined, resulted in a unit fraction of 1-half, which is evidence of the fraction as a measure meaning and how geometric similarities from the fraction as an operator meaning are connected.

Teacher C documented the following conversation they had with two children during this task, where they were exploring how many ways different a collection of cookies could be partitioned. Two children provided the following explanations. The first was Child 51, who was discussing their thinking:

> It's like if I had one person to share my cake with—just me, I would have the whole thing. More people mean I need to cut it up even, and give away…so everyone gets some [gesturing a dealing action of four parts of a circle], so I don't get it all, and I get less [gestures a shrinking action with hands].

Child 54 added, 'yeah, if I only have to share between four [people], my piece is bigger than if I had to share it [the cake] between 10…then everyone gets a skinny bit'. Child 51 nodded in agreement.

Analysing this interaction, Child 51 describes and gestures dealing the various parts of the cake, while the relationship between dividing equally to exhaust the whole (partitive division) and one of those parts (1-nth-of...) are implied, which are fraction as an operator ideas. The children recognised a distinct relationship between the number of parts generated and the impact on the size of the parts. These explanations suggested they are not only justifying that shares can be equal regardless of how many shares there are but that there is a direct relationship between the size of the share and the number of shares created. This is a fundamental idea underpinning the concept of partitioning.

The idea of unit fraction was interpreted through the use of Child's 51's gesture, where their organised structure indicated by the cookie halved vertically and horizontally suggested they were thinking strategically about how to represent four equal parts when explaining the relationship between the unit fraction generated and the size of that part—a fraction as a measure idea. Child 54's description of the task also demonstrated an understanding of the magnitude of each share if the cake were partitioned into tenths or quarters. This understanding was inferred by the child identifying the spatial dimensions of the parts (i.e., the whole thing, less, bigger, skinny), suggesting spatial language was an important component for explaining the magnitude of the parts and the relationship of these parts to the whole. This suggests fraction as a measure understanding, because they are recognising that the smaller the unit, the more units there are, but the whole as a referent has not changed.

Similarly, gesture and spatial structure were evident in the development of unit fractions and many-as-one ideas in the Summarise phase of this lesson. Three children were sharing their thinking and strategies for how many ways they could share a collection of six cookies fairly. Child 50 stated:

> If you had six groups like there [tapping in the air six times to indicate two rows of three], you'd call the parts 'sixths'—there's more [parts] though than if you just made two groups, halves…so they're smaller.

I asked Child 50 what about if you had to make four equal groups, to try eliciting their thinking about the units themselves. They used the same dotting gesture in the air, this time making a square figure, with both hands to indicate quarters: 'They would be fourths—and bigger…than if you had the sixths' (Child 50).

The child did not explicitly indicate the many-as-one idea of three cookies as the unit of 1-quarter by this statement. Nevertheless, they did provide an explanation that demonstrated a developing understanding of the name of the unit fractions and how many parts are generated, and the relationship to the magnitude or size of each part when the whole is partitioning into fourths or sixths.

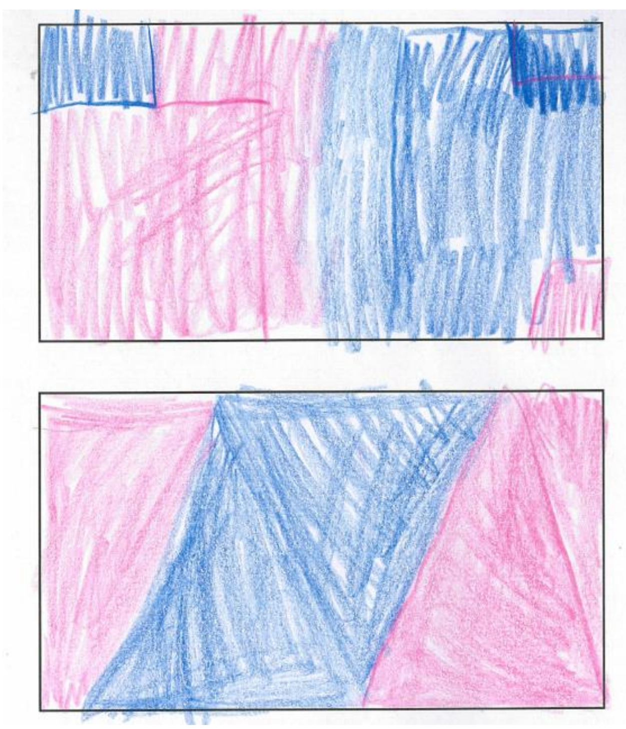
Although emergent, these examples of engaging in fraction as an operator and fraction as a measure ideas simultaneously are crucial for the development of partitioning, which is the fundamental basis of rational number knowledge (Confrey et al., 2014b; Siemon, 2003).

6.4.1.3 Lesson 7: Tablecloths

As described in Table 6.4, Class C was introduced to the tablecloth lesson in a slightly different way to how Class B engaged with it. The children's engagement with the tablecloth lesson also provided evidence of how they were using spatial visualisation in the development of fraction as a measure ideas when creating and justifying equal parts. For example, Child 67 stated, 'if you take a bit from one bit you need to put it back on the other half, but it makes the same amount'. As they spoke, they pointed to the pink triangular region on the left-hand side of their second tablecloth, shown in Figure 6.17.

Figure 6.17

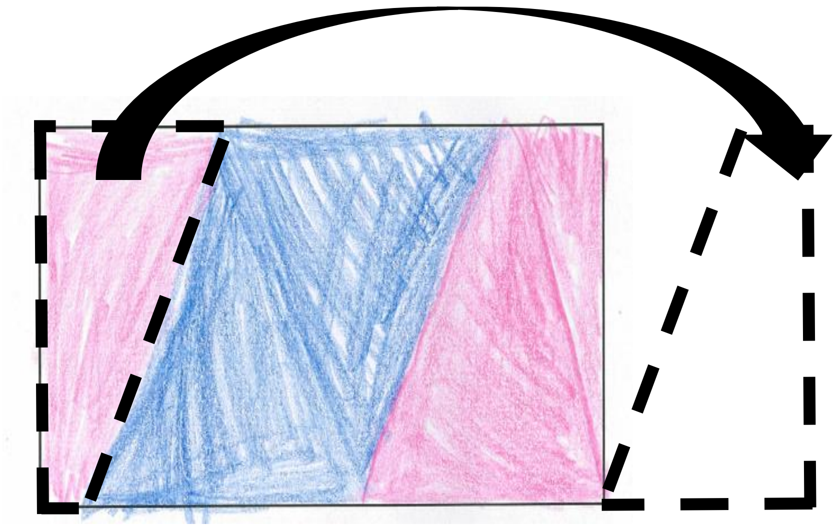
Child 67's Tablecloth Representations



I asked them if they could explain what they meant by this, and they said that if they moved the left-hand pink section to join the pink region on the right-hand side, it would ‘look the same size as the purple part’, meaning they were equal parts (see Figure 6.18).

Figure 6.18

Interpretation of Child 67’s Description of How They Mentally Moved Parts of Their Tablecloth



This child’s explanation demonstrated an awareness of and ability to justify how they determined the size and equality of each coloured part through a complex visualisation process.

6.4.2 Key Indicator: Reinitialising the Unit

The second key indicator evidenced in the teaching experiment was *reinitialising the unit.* An analysis from Lessons 3, 4 and 5 is now discussed.

6.4.2.1 Lesson 3: Visualising the Share of a Cookie

During Lesson 3, the activities were designed to build on the knowledge children had stared to develop in Lessons 1 and 2. The problem required children to think about the following context:

Imagine what a cookie would look like if we had to share between two, then four, then eight people? Can you imagine and predict what might happen to that cookie as each group of visitors arrives?

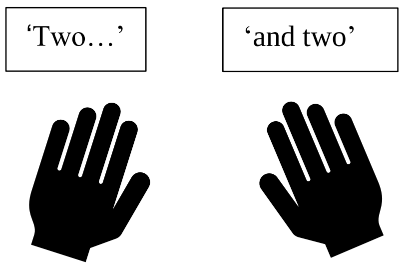
Large paper circles were provided for the children to use, but the emphasis was on spatial visualisation and communicating their predictions before they folded or drew on the circles.

Teacher C observed Child 66 working through the problem. Child 66 drew lines on their circle to partition it into quarters, explaining by gesturing with their hands a fold from top to bottom as the first half, then side to side gesture for the second half: ‘the more folds, or parts, they [the parts] get smaller’. Child 66 said that they had noticed that after they drew the circle partitioned into quarters after performing two successive folds in half, they thought it must mean each of the parts (quarters) would be split in half again if a third fold in half was performed. Teacher C asked why they thought that, and the child replied they knew that ‘the half fold meant the last two parts were cut into two [gesturing a book fold action with their hands]’, so they imagined it would be the same. That is, that every current part (quarters) would be split in halves to create eight equal parts. Child 66 drew their thinking by drawing the split of each quarter. They folded their paper cookie in front of Teacher C to check—they were so excited to find out they were right! The description suggested they were developing an appreciation of units of units (Confrey, 1994). This is an indication of the reinitialising key indicator of the local instruction theory.

This example also suggested the child was considering the structure of the whole in relation to the number of parts doubling, while the size of the parts halved through each successive fold. Teacher C asked me to model how they could extend such thinking at this point in the lesson. As a result, I asked Child 66 if they could imagine this same process with groups of cookies—like four cookies. The child stated sharing between two children, ‘[it] would be two and two’ (cookies each). As they stated ‘two’, they placed one hand on the desk in front of themselves, placing the other hand down on the desk beside the first hand as they said, ‘and two’, using their hands to represent the groups of cookies (see Figure 6.13).

Figure 6.13

Example of Child 66 Gesturing the Parts of a Discrete Set



Prompting Child 66 with the idea that two more children had arrived and now the cookies need to be shared between four, I asked them to consider the set and what 1-fourth would be. After some thinking through about what one-fourth meant (i.e., four equal parts needed to be created), they used the same gesture to represent a 2 x 2 array formation: ‘Each fourth… is one cookie each’. The next step of the problem was that the four cookies had to be shared between eight children. This step was quite challenging as there were no concrete materials explicitly provided for this task, which I had initiated upon Teacher C’s request. The child thought about the problem for several minutes, repeating statements like, ‘eight people mean they’re [the shares] eighths’. After repeating the same gesture for the previous setup of sharing between four people, Child 66 suddenly looked up and stated, ‘Oh, it’s half a cookie! Half a cookie is an eight…1-eighth’. Using the same cupping gesture as above, Child 66 explained that each cookie had to be split in half. Again, this indicated that this child is visualising the partitive division and recursive multiplication idea. That is, they are able to reassemble the whole after imagining the act of fair sharing four cookies between eight people. Further, the unit fraction idea from the fraction as a measure meaning is evident in their understating of the size of one fair share, linking the size of the share in relation to the whole.

6.4.2.2 Lessons 4 and 5

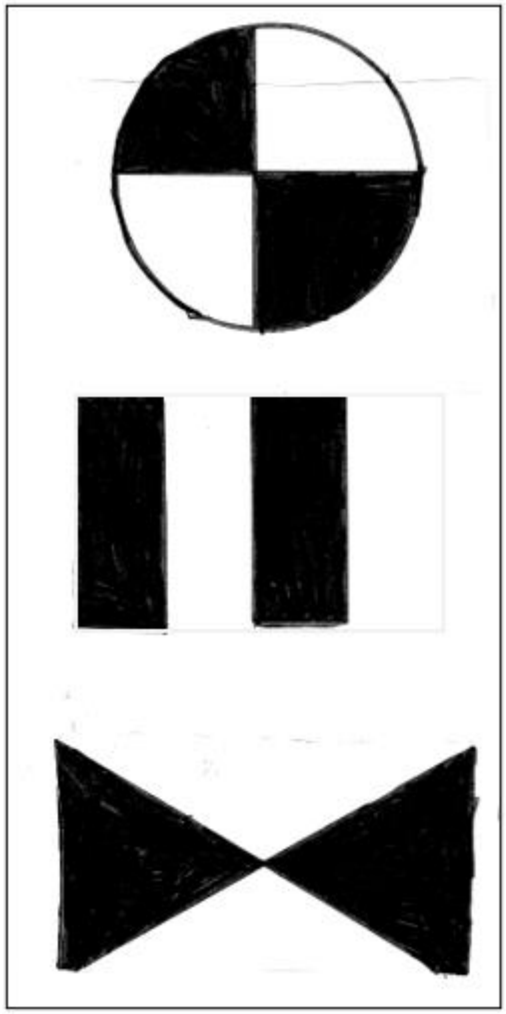
As described in Section 6.3, Lessons 4 and 5 were modified for this iteration and the content from Lesson 7 (Tablecloths) was introduced. As part of this lesson, the children were provided with a range of blank paper rectangles ('tablecloths') to design various ways equal parts of two or more colours could be represented (i.e., the tablecloth had to be half red and half green; the tablecloth will represent red, yellow, and pink equally, etc.).

The focus was on thinking about and visualising the proportions of colour, rather than the number of individual parts coloured. The following representations were provided with explanations captured by Teacher C and myself.

Child 56 stated they wanted their tablecloth to be black and white, the same colours of their favourite football team. This child explained to Teacher C that tablecloths can be different shapes, 'because you can get round and square tables in kitchens' (see Figure 6.14). They traced around a roll of masking tape to create a circular tablecloth.

Figure 6.14

Child 56's Representation of Different Tablecloth Shapes

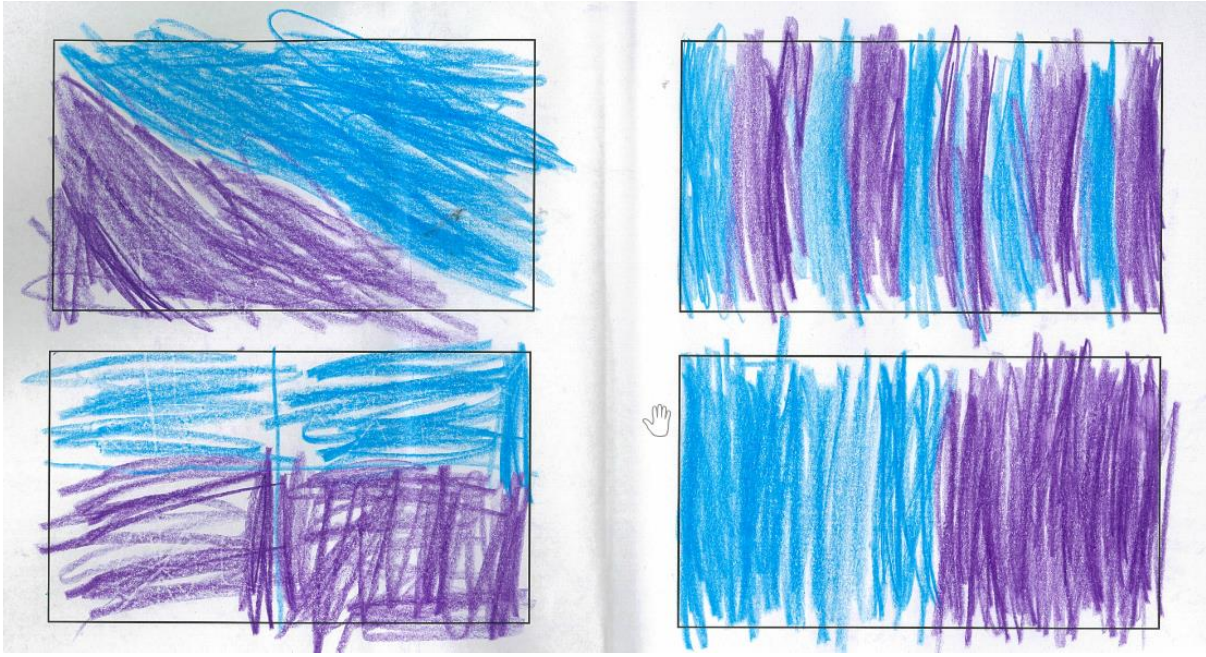


The child explained to Teacher C that even though the tablecloths are different shapes, they are all half black and white, because there are '2-fourths in each half, and 2-halves in a whole'. This is evidence of both the equal shares key indicator of the local instruction theory and emerging understanding of reinitialising the unit, where this child is considering the units of quarters within the units of halves.

Similarly, Children 58, 70 and 67 worked together. When I asked them about their representations (see Figures 6.15, 6.16 and 6.17), they concluded that all of their tablecloths represent two colours equally.

Figure 6.15

Child 58's Representation of Tablecloths

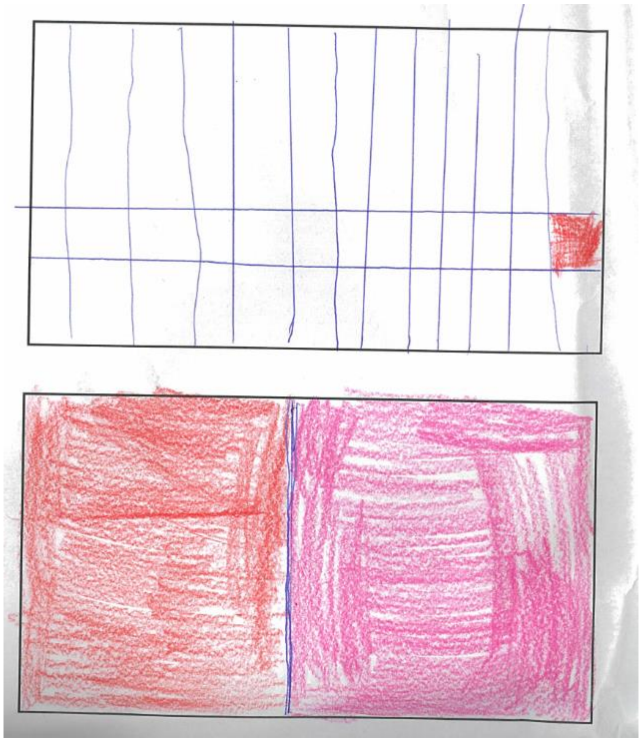


Child 58 stated they 'broke the half up and put the colours in different places', in relation to their work sample shown in Figure 6.15.

Child 70 described they were trying to make an 'even number of little squares so that the pink and red parts were the same', referring to splitting the proportion of pink and red halves equally across their tablecloth (see Figure 6.16).

Figure 6.16

Child 70's Representation of Tablecloths



When I asked them how many squares they were intending to draw for each half, Child 70 said they thought they 'might do 60 squares' and, after some thinking, stated they would need 30 of each colour. When I asked the child how they had partitioned their tablecloth, they said they had drawn lines down the middle first (two lines at 90 degrees to create quarters), then added more lines either side of these partitions to try to create a number of equal groups. Even though equal groups were not completed, this behaviour suggested this child was sensitive to the half benchmark in their attempt to create equal groups.

In summary, these examples indicated that the children were developing the fraction as a measure meaning because they were able to create a range of unit fractions and compare these in relation to the whole. For example, the suggestion that 'breaking up' or distributing the colours equally on the tables is an indication of the many-as-one idea. This idea was evident in Child's 58 explanation of ‘2-fourths in each half, and 2-halves in a whole’, and Children 58 and 67’s representation where they have created (or attempted) units of tenths and sixtieths to distribute the two colours equally. This indicates the children were demonstrating spatial proportional reasoning, because they were making comparisons between quantities (unit fractions) within each tablecloth and between tablecloths—regardless of shape or size in Child 58’s case. Further, the suggestion that ‘moving parts’, ‘breaking up’ a unit of half and redistributing suggested they were engaging in spatial visualisation to determine these proportions, as a strategy for reinitialising the unit.

6.5 Insights From Post-Tasked-Based Interview

To examine the impact of the five lessons on children’s learning, a comparison between the children’s performance on the pre- and post-TBI assessment in the form of a paired sample sign test is presented. Thematic analysis of the post-assessment TBI findings will follow.

6.5.1 Comparison of Pre- and Post-Task-Based Interview Responses

A detailed rationale for performing a paired sample sign test was provided in Chapter Three. Twenty-one children participated in this teaching experiment; however, due to the impact of COVID-19 on school attendance, only 15 children completed both the pre- and post-assessment. Therefore, the paired sample sign test was only conducted on the data of the 15 children who completed both TBIs. Each child’s response to each item on the post-test was compared to their response to the same item on the pre-test and recorded as representing a positive change, no change, or a negative change. To enable a binomial calculation, the no change and negative change responses were bundled together. For Set One, this meant that there was a total of 60 change possible responses (15 children x 4 items). For Set Two, there was a total of 75 change possible responses (15 children x 5 items). As described in Chapter Three, significance was set at $p \leq 0.33$.

Table 6.6 presents the results from the first nine items of the task-based assessment.

Table 6.6

Paired Sample Sign Test Analysis for Set One and Two (Class C)

Paired Sample Sign Test Analysis: ($p \leq 0.33$)				
Category	Assessment item(s)	Positive change +	Negative change –	*p*-value
Trusting the Count	1, 2, 3, 4	12	60	Not Significant
Place Value Ideas	5, 6, 7, 8, 9	24	62	Not Significant

Overall, there was no statistically significant change in children's responses across the whole number items of this assessment. This is not a surprising outcome, given the limited time the children had in this intervention. It is clear that more time was needed for the children to explore such ideas to determine if this intervention would have had similar results to those for the previous iteration (i.e., Class B).

Table 6.7 presents the results of the paired sample sign test by category for Set Three.

Table 6.7

Paired Sample Sign Test Analysis for Set Three (Class C)

Paired Sample Sign Test Analysis: ($p \leq 0.33$)				
Category	Assessment item(s)	Positive change +	Negative change –	*p*-value
Fraction as a measure	10, 11, 12, 21, 22, 24	52	33	<0.000
Fraction as an operator	10, 15, 16, 18, 19, 20, 23, 24	39	80	Not Significant
Fraction as a relation	13,14, 22, 23	20	48	Not Significant

Spatial Visualisation	10, 12, 15, 17, 18, 20, 22, 24	63	79	0.002
Spatial Proportional Reasoning	8, 11, 13, 19, 22, 23, 24	62	74	0.001

There was a statistically significant change in children's understanding of fraction as a measure ideas, likely due to the emphasis on fraction as a measure ideas explored in the duration of the intervention. There were no significant changes in children's fraction as an operator or fraction as a relation understandings, even though children experienced fraction as an operator ideas frequently throughout the five lessons. Fraction as a relation was not explored explicitly.

With regard to spatial reasoning constructs, the paired sample sign test revealed statistically significant improvements in both spatial visualisation and spatial proportional reasoning, despite the short timeframe. While it is well acknowledged in the literature that spatial reasoning is malleable (see Cheng & Mix, 2014; Lowrie et al., 2019; Uttal et al., 2013), these results suggest that even limited exposure to spatial training—in this case, five lessons—can have an impact on children's spatial reasoning development. However, as described in the previous chapter, while indicative, these results only provide a limited perspective on the impact of the intervention program. Therefore, qualitative insights into the shifts in children's reasoning and mathematical behaviour between pre- and post-measures are now discussed.

6.5.2 Post-Task-Based Interview Insights

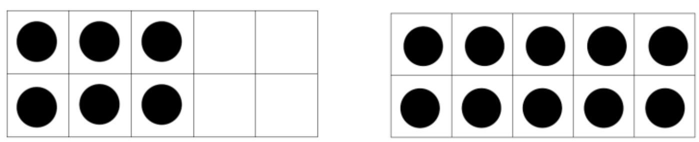
There were some differences observed in the way children communicated their thinking, with gesture and spatial visualisation and spatial structuring being prominent in the children's responses across each set.

6.5.2.1 Set One and Two Qualitative TBI Insights

With respect to Set One, the most obvious shift in the children's results is that there was a greater awareness of the arrangement and structure of objects. When children were asked to subitise collections that were structured in tens frames (not randomised), they often made spontaneous comments about noticing structures and arrangement of the dots in the post-assessment TBI. For example, the term 'half' was used by six of the 15 children in the post-TBI to describe how they subitised the collections in one or more of the cards in Figure 6.19.

Figure 6.19

Examples of Subitising Cards Provided in Set One



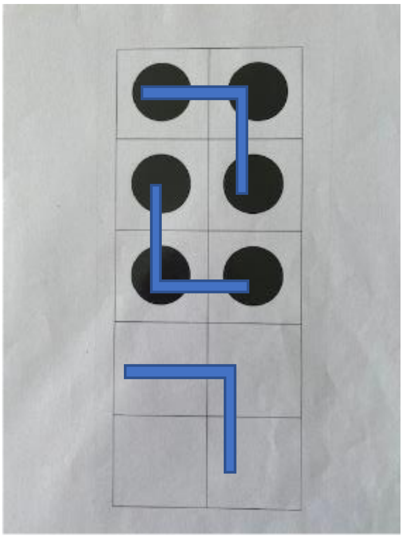
Child 69 stated they saw six as 2-halves of three dots. Child 61 stated, '10 is two rows of five. The fives are halves of the 10—just like in fractions'. These statements indicated that the children were paying explicit attention to the structure of the dots to consider their relationship between the part-whole units they were subitising. This is evidence that children can and do integrate and develop both whole and fractional understandings about number in parallel to one another.

Child 65 commented they could see thirds when solving the problem in Item 3 (Tens Frames Bananas): *There are this many bananas in a bowl, and three more are added. How many bananas are there altogether?* The children were shown a tens frame with six dots in it representing the 'bananas' as the prompt for this problem.

After Child 65 stated the correct answer of nine to the above problem (counting on from six), they immediately went on to state that they could see thirds 'within the group of nine', because each third was made up a ‘triangle of three’ dots. They traced their finger over this triangle as they described this finding—indicated by the blue lines on the stimulus provided in this item in Figure 6.20.

Figure 6.20

Item 3 Stimulus and Representation of Child 65 Gesture (Blue Lines)



This explicit attention to the arrangement of the collection suggests the child has a structural understanding of three as a composite unit and unit fraction. It was these types of behaviours—where children referred to the various spatial structures in their responses—that indicated an integrated approach between fraction understanding and whole number multiplicative structures is beneficial.

There were also differences in children’s explanations of Set Two that suggest the intervention was influential. The most notable was in Item 8 (Proportional number line task). In Item 8, the children were asked to place eight and 16 on an unpartitioned 0–20 number line, and 48, 67 and 26 on a separate, unpartitioned 0–100 number line. Twelve of the children described the number lines as having ‘more or less numbers’, even though the lines were the same length—indicating an understanding of density and the proportional nature of the representations. Of the

12 children who successfully completed this task, 10 referred to *visualising* the halfway point on the number lines and relating this to a numerical value. For example, the children stated eight and 48 were close to halfway on each line (0–20 and 0–100 number lines, respectively), and seven children used their hand to state ‘the middle’ is 10 and 50 of each line. This took the form of a sawing action with their hands (to explain they were ‘halving’ the number lines), or a measurement gesture with palms facing together, suggesting they were trying to create halves to determine where the halfway mark (10 or 50) should be partitioned. This is a shift in the level of understanding demonstrated by these children in comparison to the pre-intervention assessment phase; in the pre-assessment, those who attempted the task typically took a counting approach to positioning the numbers, with only one able to complete the 0–100 number line successfully, in comparison to seven in the post-assessment. It could be inferred that the focus on spatial proportional reasoning and spatial visualisation to determine the size of fractional parts in the tablecloths had an impact on how children engaged in thinking about quantity in this item.

6.5.2.2 Set Three Qualitative TBI Insights

As indicated in the quantitative analysis above, there was some statistically significant improvement in children’s spatial reasoning capabilities between the children’s pre- and post-intervention TBI. The following tasks illustrate the connections between spatial constructs and the fraction ideas.

Item 12 (Comparing fractions) asked children to consider, *which fraction is bigger, 1-third or 1-eighth of a whole?* In 11 of the 15 responses, a *gesture* was evident when the children described why 1-eighth was smaller than 1-third, similar to that shown in Figure 6.21.

Figure 6.21

Example Images Representing the Gesture Children Modelled in Item 12

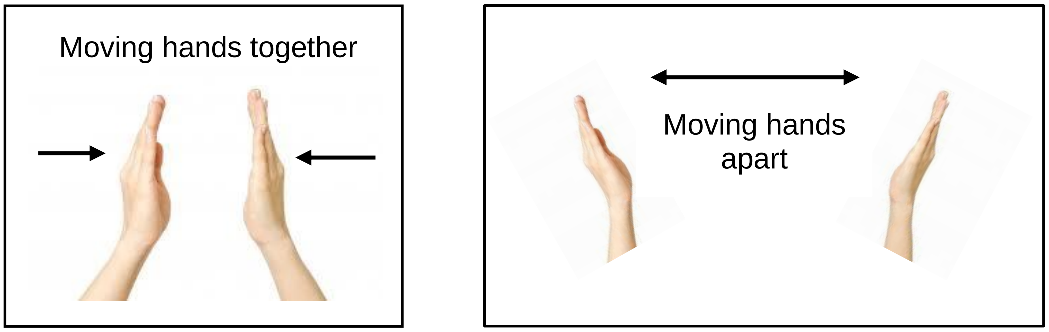


Figure 6.21 represents how children gestured a unit fraction with their hands. When describing 1-eighth, they would bring their hands closer together, indicating a smaller unit, and when describing a third, their hands would be further apart, indicating a larger unit fraction. These actions indicate an understanding of partitioning within a fraction as a measure meaning and the relationship between the number of parts and the size of parts. Several children who used this gesture would iterate the unit by moving their widely parted hands in a linear motion three times, to represent three units of measure, or with their hands closer together to represent smaller units, iterate multiple times in a linear fashion representing eighths. In the pre-assessment, no child answered that 1-third was bigger than 1-eighth, and no gestures (other than a shoulder shrug) were evident. In the post-assessment, 13 children scored a correct or partially correct answer (i.e., partially correct was scored if the child stated one-third but not able to explain why).

Item 19 (Scale the picture) revealed an improvement in children's understanding of proportion. This task required children to draw a rectangle to scale using a referent image and to describe the proportional relationship between the two images. Many children described the sides of the triangle had 'doubled in size' and then determined that all sides of the rectangle also need to double in length. For example, Child 60 stated, 'the square [rectangle] looks like it needs to be double along the sides to fit right, 'cause that's how the triangle looks. So, it means the whole thing has doubled'.

In the pre-assessment phase, only two children articulated that there was a change in size between the two images; in the post-assessment, many children who completed this task used the terms, double, half, or twice as (big/long) in their description of the relationship between the images. This suggested that although there was no statistical significance in children's fraction as an operator ideas found in the TBI analysis, there was evidence of such understating being influenced by their spatial proportional reasoning. That is, the children described that the doubling in size was in proportion to the triangle and the circle generally in their descriptions.

Similarly, Item 22 (Fred's pizza) was correctly answered by 12 children in this phase in comparison to just one in the pre-intervention TBI. To recap, the problem required the children to consider the following problem:

> *Fred is ordering a pizza that he will eat all of in one sitting. He asks the pizza man to cut the pizza into four pieces, not eight, because he says he can't eat eight slices at once. Does his request make sense?* (Adapted from Dole et al., n.d)

As there was no stimulus provided, other than an image of 'Fred', the 'cutting' or 'sawing' gesture accompanied eight children's responses. That is, the children were describing that a slice of pizza can be cut many times, 'but it still takes up the same amount of space—pizza' (Child 62). The lack of stimulus also suggests the 12 children were using spatial visualisation when the problem was explained to them, highlighting the use of this spatial construct.

The last task that showed the greatest improvement was Item 15 (Missing faces). The children were shown a picture of four smiley faces in a 2 x 2 arrangement, and they were asked: *If this is 2-thirds of the whole set, what is missing?* Of the eight children who engaged with this task (there were no correct answers in the pre-intervention assessment), most (n=6) referred to seeing ‘lines’ or ‘rows’ of faces to group and unitise. This suggest they were seeing two faces as the unit fraction of 1-third, rather than interpreting the problem as multiples of two. For example, Child 56 stated, ‘this is a third’, as they drew a ring around two faces in Item 15 to indicate the unit fraction, then, ‘The whole is missing one of these [units], so you add another third’, and the child proceeded to draw two more faces. Child 64 also stated, ‘if what’s missing is a third, then it needs to be a group of two, because you need 3-thirds to make a whole—which is a group of these [faces]’. This is the distinction between this task being interpreted as a fraction problem as opposed to whole number multiplication. This change in strategy suggests that spatial structuring (e.g., identifying patterns and regularities in the groups of faces to identify fair shares and then times-as-many these shares) was a useful tool to building flexible understandings about rational number ideas, in the context of fraction as operator.

6.6 Chapter Summary

This chapter has presented the findings from the final iteration of the teaching experiment, concluding Phase Two of this study. Although the results from this iteration are based on a shorter data collection timeframe, the findings suggest that the even limited exposure to the local instruction theory enabled children to demonstrate emerging understandings of two key indicators—*creating and justifying equal shares* in discrete and continuous models and *reinitialising the unit.*

With reference to the creation of equal parts, spatial visualisation played a key role in developing the fair share, fraction as an operator idea. Specifically, children used spatial language and gesture to explain how they visualised moving parts on top of each other or next to each other when justifying that parts would be the same or not.

The evidence presented in the analysis of the intervention provides key insights into the children's thinking. First, the relationship between spatial visualisation and the fraction as a measure meaning was evidenced throughout the first five lessons. Similarly, the fraction as an operator ideas of fair shares though partitive division was supported by children's awareness of, and attention paid to the spatial structures of the representations.

There was a relationship between spatial visualisation and the partitive division/recursive multiplication idea, as children used gesture to describe how they understood the outcome of sharing the same collection or region, between a different number of nominated shares (e.g., 12 cookies between two, four, six and eight people: partitioning a tablecloth into a different number of parts). This evidence suggests children were developing the key indicator of reinitialising the unit. Although children's collective responses were not as sophisticated as those in the previous class, this evidence does suggest that partitioning as a central concept underpinning the fraction as an operator meaning is a spatial activity.

In discrete context primarily, children's use of and attention to the spatial structures of the representations assisted them in making sense of the relationship between many-as-one, fraction as a measure idea and the fair sharing, fraction as an operator idea. The use of spatial structure in these contexts enabled children to visually compare and imagine and justify how they knew fair shares could be generated, relating to the key indicator of creating and justifying equal shares. Moreover, some of the children also used spatial structure when experimenting with the doubling and halving fraction as an operator idea to determine composite fractions (fraction as measure) and naming the magnitude of the parts, which is evidence of the key indicator reinitialising the unit. Spatial structuring was also evident in the way children engaged in partitioning discrete collections specifically, as it was associated with how children conceptualised the outcome of the share from increasing the number of shares.

The quantitative analysis of the pre- and post-assessment data revealed that the Class C children made much greater use of their spatial visualisation and spatial proportional reasoning capabilities, despite a much shorter timeframe for this teaching experiment (compared to Class B). Further, fraction as a measure items were also significantly improved from pre- to post-assessment for the 15 children. The qualitative analysis of the TBI also suggests that the focus on the geometric properties of the shapes (circles and rectangles primarily) within the intervention helped children develop a sense of fraction magnitude when estimating and justifying the outcome of partitioning.

The qualitative analysis of the post-intervention TBI revealed that spatial structure was a critical component, specifically when describing the relationship between composite units and unit fractions (i.e., three as a composite unit and as a unit fraction of nine for the tens frames bananas task, 10 as a composite unit and unit fraction of 20 for the number line task, etc.). This construct suggested it assisted children to 'see' and name quantities in different ways, which connected both whole number and fraction ideas, even though there were no statically significant improvements in their whole number knowledge for Set One and Two.

The children's engagement with spatial visualisation and spatial proportional reasoning was evident in far more children's responses than the pre-assessment TBI for items that had no visual stimulus—such as Items 12 (Comparing unit fractions) and 22 (Fred's pizza). This assumption is made based on the accompanying gestures that children typically used when explaining and justifying their thinking, which suggested they were visualising different fraction measures to compare and justify their reasoning.

Finally, gesture again was a representation observed as children communicated their ideas about creating quantity (fraction as operator) and magnitude (fraction as measure). The use of gesture was strongly associated with how children were explaining the outcome of partitioning in various contexts and suggested they were associated with the predictive outcomes of determining the different sizes of shares. That is, the process of creating various parts was indicated by children’s gestures (such as cutting, pulling apart or distributing imagined shares as a fraction as an operator process), and the children also used gesture when justifying the size of parts created (fraction as measure).

Chapter Seven will move this thesis into Phase Three: The Retrospective Analysis, to interpret these findings and present a cohesive discussion of this study contribution to young children’s fraction development.

Chapter 7: Discussion

7.1 Chapter Overview

This chapter moves the thesis into the third phase of the Design-Based Research (DBR) study – the retrospective analysis phase. The purpose of this phase is to place what the children have learned and how they developed their knowledge in a 'broader theoretical context by framing it as a paradigmatic case of a more encompassing phenomenon' (Cobb & Gravemeijer, 2104, p. 83). That is, what the findings suggest for the teaching and learning of fractions in the early years of schooling. Section 7.2 provides a brief review of the study, its aims, and design. Sections 7.3 and 7.4 answer the research questions by exploring the extent to which children's fraction and whole number knowledge improved as a result of the intervention, and how spatial reasoning supported this learning. The significance of the study is considered in terms of current theories about the teaching and learning of fractions in the early years, the role of spatial reasoning in supporting young children's learning of fractions, and the suitability of the local instruction theory in section 7.5. The chapter concludes with a summary of the key outcomes of this study (section 7.6).

7.2 Review of the Study

This study was prompted by the persistent difficulties experienced by primary school children when learning fractions (Callingham & Siemon, 2021; Thomson et al., 2020; Yearly & Bruce, 2014). To examine this phenomenon, a Design-Based Research (DBR) study was implemented to develop and refine a local instruction theory for teaching fractions in the early years of primary school. The purpose of local instruction theories is to explore and refine current theoretical perspectives about a specific mathematical topic, through the design, implementation, and analysis of teaching experiments (Prediger et al., 2015).

This study was developed in three phases, which are summarised below.

7.2.1 Phase One: Preparation

The purpose of this phase was to conjecture what the local instruction theory may entail for the targeted age group, informed by an extensive transdisciplinary literature review. A range of perspectives from the fields of neuroscience, educational psychology and mathematics education were considered to examine children's potential for learning a range of different fraction meanings and to determine ways of reasoning that may support this learning. Fractions has been the subject of 'rigorous and knowledge yielding study in mathematics education for decades' (Bruce et al., 2017, p. 156) yet children are still demonstrating persistent difficulties in this area of mathematics. Thus, a transdisciplinary examination of theoretical perspectives concerning the complexities of learning mathematics enables the 'relatively untapped research area of spatial reasoning' (Bruce et al., 2017 p. 156) to be explored in relation to the development of fractions. Not only does a transdisciplinary approach to investigating this problem aim to improve the educational outcomes for young children, but it ultimately helps bridge the knowledge gap that is evident between these disciplines (Bruce et al., 2017).

Current approaches to the teaching and learning of fractions in primary school were also examined in light of the theoretical perspectives, to consider how the conjectured local instruction theory would inform an innovative approach for learning fractions. It appeared that a part-whole emphasis and counting based approaches are not only common in instruction, but significantly limit children's understanding of fractions and their ability to establish flexible, multiplicative ways of thinking and working with them (Kieren, 1988; Lamon 2006; Gould et al., 2006). Confrey's (2008) splitting perspective was chosen as a basis for this study, where equipartitioning is considered as the foundation for all rational number knowledge and develops in parallel to counting (Confrey 2012; Confrey & Maloney, 2015; Confrey & Scarano, 1995). Through their extensive work, Confrey and colleagues (2014b) developed the learning trajectories for rational number reasoning, which are described as a *framework for understanding* rational number (Confrey et al., 2010; Sztajn et al., 2012). As described in Chapter Two, this framework demonstrates how each of the seven trajectories are connected to, and develops from, the foundational Equi-Partitioning Learning Trajectory (EPLT) (Confrey et al., 2014b; Confrey & Maloney, 2010).

Another reason this perspective was chosen as a basis for informing the local instruction theory is that Confrey et al., (2014b) suggests that their learning trajectory framework represents three distinct meanings of fractions: fraction as an operator, fraction as a measure and fraction as a relation. However, it appeared that spatial reasoning could play a substantial role in the development of the early ideas associated with the fraction meanings described in this framework. This conjecture is consistent with several studies suggesting it plays an important role in young children's mathematical development more generally (e.g., Bruce et al., 2016; Matthews & Ziols, 2019; Mulligan, 2015; Mulligan et al., 2020).

The local instruction theory consisted of a series of key indicators to describe the types of fraction meanings and associated ideas children were to develop, and a description of how a spatial reasoning approach supported their learning. A suite of tasks was designed in relation to each of these key indicators and trialled with a participating Year 2 class. The analysis of the children's responses in the preparation phase, enabled new insights to be considered for the teaching experiments. That is, the children's spontaneous use of gesture provided an interpretation of how they visualised, and mentally manipulated, various shapes and objects when describing their fraction understanding. In addition, the children's use of spatial language provided another lens to analyse and interpret how they were reasoning with various representations. These two forms of representations were added to the codebook for the thematic analysis.

The analysis of the children's responses to the tasks also enabled the key indicators to be refined during this phase, and the full intervention program to be sequenced in readiness for the teaching experiments. As described in Chapter Four, the tasks enabled children to explore connections between discrete and continuous contexts together through the mental act of splitting. This meant the first two conjectured key indicators were combined. In addition, the tasks enabled the children to develop more sophisticated understanding of fractions than initially anticipated in the key indicators, through their ability to recognise proportionally equivalent fractions. This meant the third key indicator was redeveloped to better reflect their learning potential. As a result, the local instruction theory was refined to four key indicators and is re-presented in Table 7.1

Table 7.1

The Local Instruction Theory (Version Three)

Key Indicators	Characteristics of Tasks	
	Primary Fraction Foci	**Spatial Reasoning Approach**
Creating and justifying equal shares	**Fraction as an Operator:** Fair shares Doubling/ halving Partitive division/ recursive multiplication, Geometric symmetries, Similarity **Fraction as a Measure:** Many as one, Measure, Composite units, Unit fraction	Visual perception of equal groups (drawing on spatial structures and arrangements). Equality of parts regardless of model (i.e., equal parts for discrete collections and continuous models less than and greater than 1). Visual awareness of the geometric properties of parts and sets (e.g., shape, orientation, pattern, symmetry). Observing the physical transformations of partitioning (dividing and reassembling), and visualising and

		predicting the outcome of a nominated split (e.g., spatial visualisation)
Reinitialising the unit	**Fraction as a Measure:** Composite units, Unit fractions, Part-whole fractions, Equivalent fractions **Fraction as an Operator:** Fair shares, Doubling /halving; Partitive division/ recursive multiplication, Times-as-many, Similarity **Fraction as a Relation** Many-to-one Distribution	Visualising measures between parts and wholes, and between composite and unit fractions through unitising. Exploring the spatial structure and arrangement of objects and sets to create and compare different unit fractions. Visualising magnitude relations between parts (double/ half/ times as many) the distribution of parts to determine equivalence.
Recognising proportional equivalence	**Fraction as a Relation** Distribution, Proto-ratio, Equipartitioning multiple wholes, **Fraction as an Operator** Doubling/ halving, Times-as-many, 1-nth-of..., Scaling, Geometric symmetries, Similarity **Fraction as a Measure** Composite units Unit fractions Equivalent fractions	Visualising the relationship between equivalent measures, of same and different wholes.
Connecting multiplicative relations	**Fraction as a Relation** Many-to-one, Distribution, Proto-ratio **Fraction as an Operator** Doubling /halving	Early relational understandings between the structure of part-part and part-whole quantities. (e.g., visualising and justifying the relational magnitude of fractions in relation to other fractions (e.g., 1-quarter is a half of a half/ twice as

	Partitive division/ recursive multiplication Times-as-many, 1-nth-of..., Scaling **Fraction as a Measure** Composite units Part-whole fractions Equivalent fractions	small); and working flexibly with non-symbolic simple ratios (e.g., 1:2 = 2:4).

7.2.2 Phase Two: Teaching Experiment

To explore the viability of the local instruction theory as a framework for developing an extended range of fraction meanings through a spatial reasoning approach, the intervention program was implemented in two additional junior primary classes. In preparation for each teaching experiment, the individual classes were observed during their typical mathematics lessons prior to the intervention. The purpose of this was to gain insights into the general classroom environment, pedagogical styles enacted by the teacher, and cognitive behaviours exhibited by the children. In addition, a pre- and post-intervention Task Based Interview (TBI) was conducted with each participating child immediately before and after the intervention. The pre-intervention TBI data was used to gain further insights into their ways of thinking and reasoning about assumed familiar mathematical content (i.e., whole number ideas) and likely unfamiliar content (i.e., the three fraction meanings). The pre-intervention TBI for Class B revealed that spatial structuring was an additional spatial reasoning construct that some of the children appeared to utilise in the interview. After considering this construct within the current literature as part of the micro analysis and refinement conducted within DBR studies (Prediger et al. 2015), the decision was made to include spatial structuring in the intervention program and codebook for thematic analysis.

This information was used in addition to the classroom observations, to consider if any adaptions might be needed in the specific tasks within the intervention program. As reported in Chapter Six, the complexity of learning difficulties Class C demonstrated during this period resulted in minor changes to the focus and sequence of the tasks for their teaching experiment.

Only one class experienced the full intervention program as the study was interrupted by the COVID-19 pandemic, and associated restrictions. Despite this limitation, the two teaching experiments provided powerful evidence that the children were able to develop a sophisticated understanding of the three meanings of fractions, when experienced through a spatial reasoning approach. Additionally, the post-intervention TBI data confirmed that various levels of improvements in children's whole number, fractions, and spatial reasoning capabilities had occurred as a result of the intervention.

7.2.3 Phase Three: Retrospective Analysis

The third phase of this study answers the research questions and support the discussion of the significance and contribution to early childhood mathematics education. The purpose of the retrospective analysis phase is to discuss and interpret what has been learned from implementing the local instruction theory through the intervention program. Specifically, the ways in which children demonstrated an understanding of the three meanings of fractions through explicit spatial reasoning constructs, and to compare these findings to what is currently known about young children's fraction capabilities.

To begin the retrospective analysis phase of the study, the two research questions will be answered. The first question will discuss the extent to which children developed an understanding of each of the three meanings of fractions, evidenced by the associated fraction ideas, and how particular spatial reasoning constructs supported this understanding. The second question will examine the extent to which the children's whole number knowledge was influenced by this approach.

7.3 Research Question One

To what extent and in what ways can young children demonstrate an understanding of an extended range of fraction ideas experienced through a spatial reasoning approach?

As described in previous chapters, each lesson of the intervention was informed by a relevant key indicator of the local instruction theory. Each key indicator included opportunities for the children to explore multiple meanings of fractions in relation to the intended learning goal. The following section will discuss the depth of understanding the children demonstrated for each of the fraction meanings in relation to the key indicator that best highlighted their learning.

The children's understanding of the fraction as an operator meaning will be discussed in relation to the key indicator of *creating and justifying equal parts*, as it was foundational to the early development of the partitioning concept. The children's ability to work with the fraction as a measure meaning will be highlighted through the key indicators of *reinitialising the unit* and *recognising proportional equivalence*. Finally, the children's understanding of the fraction as a relation meaning will be discussed by the children's success with key indicator of *connecting multiplicative relations* and the post-intervention TBI.

7.3.1 Fraction as an Operator

The most important finding from this intervention in relation to this fraction meaning, is that the children understood the relationship between the number of parts and the size of the resulting parts, which is the basis of the partitioning concept (Lamon, 2020). The local instruction theory included ideas from the fraction as an operator meaning in all four key indicators. However, in relation to this fraction meaning, the key indicator of creating and justifying equal shares provides the most salient evidence for discussing children's understanding of the fraction as an operator meaning, and how spatial reasoning influenced this development.

7.3.1.1 Creating and Justifying Equal Shares

This key indicator was determined as the starting point of the local instruction theory based on literature that suggested children could create fair shares of objects and sets from an early age (Confrey, 2008, Lamon, 2007; Empson, 1999; Mamede & Nunes, 2008). However, the literature reported an overemphasis on counting parts, in early instructional approaches (e.g., Amarto, 2005; Clarke et al., 2006; Gould, 2011; Norton & Hackenberg, 2010). Appreciating the relationship between the number of parts, their size, and how they are named, has been noted as a consistent area of difficulty in children's understanding of fractions (Gould, 2013; Siemon, 2003/ 2019). As this relationship is founded on the basis of a multiplicative, rather than additive understanding, a different approach to traditional instruction was needed for children to develop the first key indicator. The following sections describe how the emphasis on spatial reasoning helped children develop an appreciation of early fraction as an operator ideas in discrete and continuous models within this key indicator.

Continuous Contexts. In exploring continuous models, the act of repeated halving via folding pieces of paper (in the form of circles, rectangles, squares, and triangles) was the basis for children's initial partitioning activity. The folding process specifically drew on the fair share, and geometric symmetries, and similarity ideas, within the fraction as an operator meaning. This focus enabled children to see the physical transformations of the shapes and parts as they occurred and allowed the size of parts to be considered in relation to the number of parts created. The children were required to pay explicit attention to the geometric properties of the paper shape being folded to observe how equal parts were generated, and how the idea of doubling the number of parts halved the size of the parts. As children experienced physically folding and manipulating continuous models (including magnetic fraction circles), they were asked to start visualising and predicting the outcome of creating equal shares.

The children commonly referred to 'seeing' or 'imagining' (interpreted as visualising) the outcome of creating equal parts. This allowed them to accurately predict the transformations that would need to occur to create multiple shares. For example, in Lesson 5 (Cookie fraction estimation), the children were presented with images of cookies that had a segment 'eaten' by a mouse, and they were asked to justify how much of the cookie it had eaten. In Chapter Five, the following observations exemplified the connection between identifying fair shares, and the role spatial visualisation played. First, Child 47 explained their thinking about a cookie that had 1-third eaten: 'We can see that the chunk is like a third because in our head, if you move this piece [the missing third] around, you'd get another two of the thirds to cover the whole cookie'. While this child was explaining their thinking, both them and their partner were observed moving their hands like they were turning a dial. This gesture suggested they were visualising the rotation of the missing third around the circle to determine there were three equal parts. They were paying attention to the geometric properties of the parts in relation to the whole circle, however conceiving and visualising the proportions of the cookie as a split of thirds.

Similarly in Chapter Six, Child 67 and 63 indicated they had visualised stacking parts of a cookie (that had been cut vertically into four pieces) to justify that the parts were not equal (see section 6.4.1.2). In addition, they described how they visualised joining these parts in ways to create equal shares, again paying attention to how the creation of parts affected the size and shape of the object. These examples illustrate the influence spatial visualisation had on children's ability to imagine and justify how many equal parts (i.e., fair shares) could be represented in this cookie.

Spatial visualisation was also associated with the doubling and halving idea in developing the fraction as an operator meaning. Child 38's example was indicative of this connection when describing how much of a cookie had been eaten with only 1-quarter of a cookie visible:

> I imagined folding this part over [signalling a mirror image flip of the quarter with their hand] to make a half, and then, you know, the other side that makes a half is two perfect quarters—so the missing part has to be three of these quarters.

This example shows the child has a sophisticated understanding of how the doubling and halving idea supports the creation of fair shares by visualising the relationship between the parts and their size in relation to the whole. It suggests the child understood the process of repeated halving and the multiplicative nature of creating composite units in this context because they envisioned the whole simultaneously with the unit fraction in the reassembly process.

Even when children interpreted the sharing tasks differently to what was intended (see Chapter Six, Section 6.4), it was evident that the emphasis on visualising and predicting the outcome of doubling and halving enabled children to develop an understanding of the relationship between the size of the parts, and the number of parts generated. For example, Child 56 represented the outcome of repeatedly halving a single cookie, drawing three cookies partitioned in halves, quarters, and eighths, respectively. I asked the child to tell me more about their representation, including how they had labelled some of the parts. They replied, 'It's doubled [pause]…No, it's actually halves. It goes from half to fourths to eighths. See, half of a half is fourths, half all the quarters you get eighths [pointing from one cookie to the next]'. When prompting the child to visualise and justify what would come next if they continued to partition another cookie, they stated, 'Ummm, that's double eight… but half of each eighth [running fingers over the cookie partitioned in eighths] ...ahh…[pause]… sixteenths!'

This example demonstrates that the child recognises the geometric similarity of the resulting part, as a split of a split (Confrey & Smith, 1995); that is, double means ‘twice as big’ and half is ‘two times as small’. As Lamon (2020) describes, this type of thinking is beyond simply considering a selection of parts in comparison to its whole, or the part-whole meaning of fractions. Rather, it demonstrates an early appreciation of the multiplicative relationship between the number of parts and the size of the parts within the halving family —that developed by visualising the outcome of repeated halving.

Several researchers (e.g., Confrey & Maloney, 2010; Pothier & Sawada, 1983) suggest that it is not just number properties for fractions that impact children’s conceptualisation of fair sharing, it is also the geometric qualities of the materials and representations that are essential for children to explore in such activities. This evidence demonstrates that utilising spatial visualisation in the repeated halving of geometric models, is a powerful way for children to develop early partitioning generalisations; that is the multiplicative nature of *a* divided by *b* shares results in a share of *a/b*. This is a critical finding as, according to the current literature, anticipating the outcome of partitioning, and recognising the multiplicative foundations of fractions, are understandings that many older children fail to comprehend (Callingham & Siemon, 2021; Siemon, 2016; Thomson et al., 2020).

Discrete Contexts. The evidence presented in previous chapters showed that young children develop sophisticated fraction as an operator understandings in discrete contexts. This understanding was clearly supported by a focus on creating and understanding the spatial structure of parts generated in the concrete representations, through partitive division and 1 nth of ideas. The partitive division idea is the process of creating fair shares through starting with the dividend and distributing a collection one at a time, until the collection is exhausted. The 1 nth of idea is complex in discrete collections, as children will often confuse the number of individual objects within a share, with the number of shares. For example, early in Class C's teaching experiment, some children confused creating groups of two cookies (from a set of eight) as sharing between two people (see section 6.4.1.1).

The 1 nth of idea is closely related to the fraction as a measure meaning in discrete contexts, as it requires children to consider many objects as one part (many-as-one)—such as four cookies is one-third of 12 cookies. This study found that spatial structuring, the process of assigning patterns and structures in representations either physically or imagined, was critical in enabling the children to make judgements about the quantities represented. For example, the children's pictorial representations described in Chapter Five, clearly portrayed a regularity and structure when recording fair shares of 12 cookies. It appeared the children were referring to the geometrical structure of parts (developed from their experiences with continuous models), which in turn, supported their ability to justify equal shares in discrete contexts. A common example was Child 45's representation, where they arranged three groups of four cookies (counters) in a 2 x 2 format, describing the set of 12 cookies as 'three squares' of four cookies. Other children demonstrated similar success with the partitive division idea, when encouraged to line their cookies in equal rows or columns. This demonstrates that a focus on the structural arrangement of the counters enabled children to recognise that one share of 12 cookies (1 nth of) is four cookies, and that the shares are equal because the structure and geometry of each share is identical.

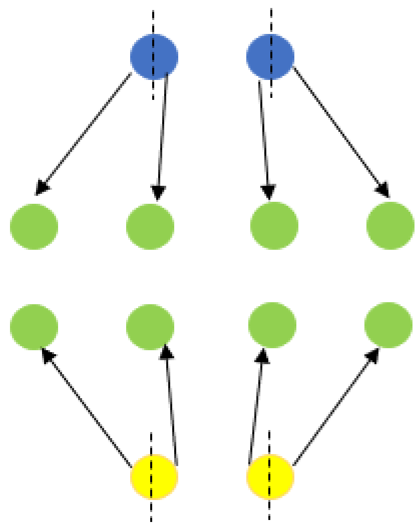
In Confrey et al.'s (2014b) framework derived from a synthesis of the literature, they suggest children first develop the fair sharing idea through 'Case A', discrete context, which is 'sharing *mn* objects fairly among *n* children, where *m* and *n* are natural numbers' (Confrey, 2012, p. 161). Children then progress to 'Case B', sharing a single whole between many (a continuous context); and finally, to 'Case C', sharing multiple wholes between *n* children to explore proper and improper fractions. They acknowledge that this trajectory of case sharing is largely conjectured from studies that explore one specific case at a time, as no study to date has explored the development of the three cases of sharing simultaneously (Confrey, 2012). While the present study has not specifically followed the development of the three cases per se, the findings do suggest that by focusing on geometric and spatial structures of the sharing contexts, enables children to develop flexible connections and generalisations about the fraction as operator meaning, in both discrete and continuous contexts.

The role of gesture was also found to be important in children's reasoning about discrete contexts. While spontaneous gestures were used by the children early in their descriptions of how they would physically partition various arrangements of discrete objects, it appeared that an emphasis on spatial structure in the intervention was also reflected in gestures. Their gestures were predominately classified as iconic, that is, those closely associated with the meaning of the idea being explored (explained in Chapter Two). However, it was the geometric attributes (gesturing the sharing of discrete sets in square-like arrangements) or gesturing row-and-column-like structures accompanying their description of fair shares that was evident.

A further example of the connection between children's gestures and spatial structure was sharing 12 cookies between eight people, which was a complex task given the age of the children and the timing of the activity within the intervention. Here, children created row-and-column-like structures after dealing one cookie to eight people (in two rows of four) and placed the four remaining cookies in a symmetrical arrangement to describe how they would split the remaining four cookies equally among eight people, as illustrated originally in Figure 6.11 (reproduced here for convenience in Figure 7.1).

Figure 7.1

Arrangement of Representation from Children 52, 55 and 57 and Their Description of Visualising the Shares



Note. The counters have been coloured to assist the discussion.

It was their accompanying gestures that enabled them to communicate how they knew each person received a fair share, as highlighted in their explanation:

> Child 52: This one cookie [referring to one of the two cookies represented in yellow in Figure 7.2] gets broken in half and shared to these two people [pulls hands apart, suggesting they are dealing out the two parts of the yellow cookie, and placing the parts on top of two green cookies], and same with this one [gestures the same partition for the two blue cookies].

While this task was difficult for the majority of the children in the teaching experiment, the ways in which some children engaged with it suggests that focusing on spatial structures in discrete representations promotes early multiplicative understandings. The children appeared to visualise partitioning the collection in a way that enabled them to construct composite units (such as eight units of one cookie and eight units of half a cookie to complete the fair share of eight shares of 1-and-a-half cookies). Several studies (e.g., Battista & Clements, 1996; Mulligan et al., 2005; Mulligan & Mitchelmore, 2013; Van Nes & Van Eerde, 2010) have found that when children were taught how to use and develop spatial structures for working with whole number collections (e.g., arranging and visualising a collection of six items as 3 x 3 structure), they had greater success in developing early arithmetic understandings of whole number. However, no studies appear to have examined the way spatial structuring can support the explicit development of the different fraction meanings, specifically the fraction as an operator meaning. Although emergent, this type of thinking cannot be achieved through an additive approach to partitioning (e.g., creating a unit fraction and iterating), because it relies on the creation of fair shares through partitive division. A critical finding from the present study is the spatial structures are influential on children's understanding of the fraction as an operator meaning, not only in the physical and pictorial representations of equal sharing contexts, but also as communicated through their gestures.

Taken together, the results from children creating and justifying equal shares thorough emphasising a spatial reasoning approach suggests that it enables children to develop generalisations about the Case A – C sharing contexts. This is an important finding, as Nunes and Bryant (2007) state that while children are 'relatively good at thinking about the consequences of sharing' (p. 4), children between the ages of 5 – 7 years are 'very bad at partitioning wholes into equal parts' (p. 4). The present study contradicts this last point, finding that through a focus on visualising the transformation between the number of parts created through doubling and halving, and partitive division, the children were able to comprehend the outcome of familiar and unfamiliar sharing situations. Likewise, the emphasis on the geometric similarity of parts in continuous contexts appears to have had an influence on how children understand and conceptualise discrete sharing contexts. That is, the use of spatial structures created, visualised, and communicated in part via the use of gesture within these sets. This understanding did not rely on counting, but a partitive division understanding of the relationship between the number of parts and their relative size. This too is an important finding, as the ability to draw connections between continuous and discrete contexts is something that Confrey and Maloney (2010) state is a complex but important milestone in early fraction understanding, as these ideas require different cognitive demands. Children need to recognise that representing fair shares of a continuous model is the formation of multiple, connecting parts. The discrete model involves the need to perceive a unit within a larger set, while understanding that the two models represent proportionally equivalent shares. This study shows that spatial reasoning is a powerful vehicle for understanding early fraction as an operator ideas that are explicitly connected to the concept of partitioning in both continuous and discrete contexts.

7.3.2 Fraction as a Measure

The children demonstrated fraction as a measure understanding in two main ways. The first was the children’s ability to recognise fractions as composite units and equivalent fractions. The children’s understanding of these ideas was most noticeable in the lessons related to the key indicator of reinitialising the unit, which is discussed in section 7.3.2.1. Secondly, the children demonstrated an ability to compare the size of fractions between unlike wholes, demonstrating proportional equivalence. The children’s responses to the lessons associated with the key indicator of recognising proportional equivalence is discussed in section 7.3.2.2.

7.3.2.1 Reinitialising the Unit

The intention of this key indicator was for children to understand that many equal parts can be named as a unit of the whole (the composite units and equivalent fraction ideas). In addition, children were provided opportunities to recognise composite units for simple, common fractions (e.g., '1' can be represented as 2-quarters and 1-half), and to explore the equivalent fractions idea, which is a generalised understanding of how fractions can be infinitely renamed.

Evidence of children's understanding of the fraction as a measure meaning is illustrated by Child 43's exploration of Lesson 5 (Cookie fraction estimation) in Chapter Five. When using the magnetic fraction circles (referring to one partitioned into tenths), this child stated, 'If five of those pieces covered a half, then it is the right piece—because half of 10 is five. I just doubled and flipped it over in my head to work out the right fraction'.

It is evident that the child is developing the composite unit idea from the fraction as a measure meaning, intertwined with the doubling and halving idea from the fraction as an operator meaning discussed above. The child's description of 'flipping in their head' is a spatial transformation term, suggesting spatial visualisation assisted in their justification for reinitialising 5-tenths as 1-half.

Other references to spatial visualisation interpreted from children's use of spatial language was evident in explanations that includes spatial transformation terms like slide, move, flip, and turn. These words were used to describe how the children described mentally manipulating the parts that were distributed in representation, particular in the Tablecloth lessons. For example, Child 30 commented:

> You can have something [gesturing parts of the third of the tablecloth] that has lines all over it, and all different shapes, but it's still a whole, and you can still make a half or a fourth if you look inside these patterns and move them in your head.

These statements indicated that many children were developing the composite unit and equivalent fraction ideas by engaging in spatial visualisation, which allowed them to mentally move and transform parts of the tablecloth to make a judgement on the relative proportions of different colours. What these understandings of fraction as a measure ideas represent, are powerful demonstrations of quantitative equivalence understanding facilitated by spatial visualisation.

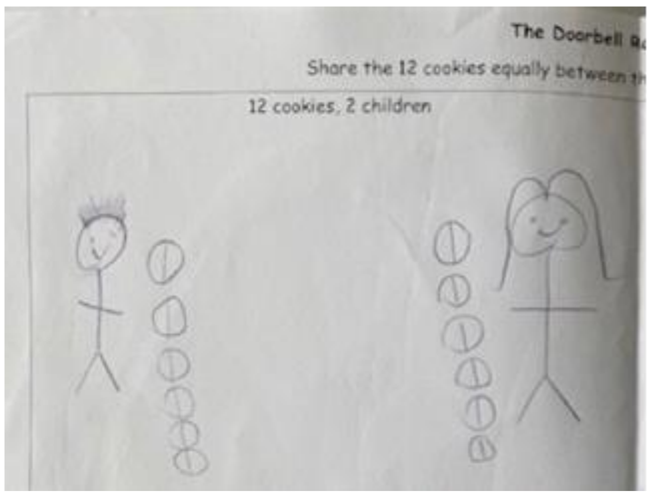
Another common observation of children's exploration of the composite unit and equivalent unit ideas for the fraction as a measure meaning, was children's use of the spatial structures within the models used. For example, the hexagon, trapezium, and triangle pattern blocks in the Pattern Block Fractions lessons enabled many children to create models that represented the relationship between halves and wholes by their symmetrical, geometric structural relationships. For example, Child 34 stated 'you can have six of these [triangles] as the same [shape and size] as the third [hexagon]'. Similarity, as reported in Chapter Five, the description of the geometrical structure and repetition created with the orange square pattern blocks (see Figure 5.17) represents the understanding of composite unit and equivalent fraction ideas. For example, Child 27 referred to a row of squares as one fourth (many-as-one and unit fraction idea), whilst Child 49 commented on how they could see 'squares within bigger squares' (emerging composite unit idea). This understanding was supported by the doubling and halving idea from the fraction as an operator meaning, however, it demonstrates the relationship between spatial structure, and the children's ability to name units of units as fraction as a measure understanding.

Other examples of how spatial structure supported the fraction as a measure ideas were evident in discrete contexts. For example, when children explored how 12 cookies could be shared fairly resulted in children looking for pattern, repetition and similarity when arranging or representing their units. They would often see row-and-column-like structures to not only determine they had equal shares, but to describe the quantity based on these structures that were created through the partitive division approach. This thinking was exemplified by the children working with me in a small group setting (see Chapter Five) on how to share six cookies fairly.

One child commented on seeing groups of ‘two, two and two’, and lines of ‘three and three’ as units of six, when promoted to name the collection by looking at how the set was arranged. Additionally, Child 62’s representation of sharing 12 cookies between two people demonstrated how spatial structure influenced how they were able to name the composite units and equivalent fractions of the set (originally introduced as Figure 6.8, reproduced below as Figure 7.2).

Figure 7.2

Child 62’s Work Sample



Child 62: When I drew the two different groups, I knew each kid would get six—a half of 12 is six. But then I just cut each cookie in half too, because you can see [using hand perpendicular to the page] …12-halves here for this person, and same for this one—two rows of halves [using hand to gesture linear groups of partitioned cookies].

The way the child explained the connection between the two rows of 6-halves as being the same as six cookies – highlights a complex understanding of the fraction as a measure meaning (e.g., composite units and equivalent fractions). What is important about these contexts, is the children were not just seeing the parts as separate, countable items, they were able to use the structure of discrete sets to unitise and ‘see’ different quantities within a defined whole. This study demonstrates that spatial structuring supports a true multiplicative foundation to naming and renaming shares. That is, the shares are created simultaneously, therefore ‘one share does not exist unless all shares exist, and each is the same 1-nth of the whole’ (Corely, 2013, p. 20).

Children across both teaching experiments provided additional examples of their fraction as a measure understanding and how spatial proportional reasoning supported this learning. Similar to Child 30’s comment above, Child 52’s comment indicated this type of thinking: ‘You can make your own lines for the parts because it’s still coloured in the half of the tablecloth’. The reference to ‘making your own lines’ suggested the child was creating units within units, appreciating the equivalent fractions in this context and utilising spatial proportional reasoning to justify halves. Similarly, Child 70 stated they needed to represent an ‘even number of little squares so that the pink and red parts were the same’, referring to the equality of the pink and red regions across their tablecloth as unit fractions of half, rather than the number of individual parts. These examples demonstrate how spatial proportional reasoning in the development of the equivalent fractions idea (fraction as a measure meaning) is also connected to the distribution idea from the fraction as a relation meaning. These are important findings because it provides further evidence for how emphasising the connection between the fraction meanings through spatial reasoning supports a relational understanding of the quantities. A critical foundation needed to work multiplicatively with fractions.

These examples also demonstrate the tightly connected nature of spatial reasoning constructs (i.e., between children’s spatial visualisation, and spatial structuring strategies) and how it can be problematic to separate exactly which construct a child may have been using, and at which point. Regardless of this, the children developed these fraction ideas on the basis of understanding magnitude through reasoning spatially, not by relying on their whole number knowledge such as counting individual parts. As described in Chapter Two, Pedersen and Bjerre

(2021) provide two novel conceptions of the quantitative equivalence concept of fractions that have been largely overlooked in the literature: unit equivalence and proportional equivalence. In the present study, it was revealed that by engaging in activities related to this key indicator, children were able to not only work with units of units in the naming and measuring of fractions, but they also developed an understanding of unit equivalence (i.e., recognising composite units and equivalent fractions within like wholes).The children's reasoning about this fraction meaning indicates they are starting to appreciate that fractions, as numbers, are infinitely divisible (Siegler et al., 2013) and therefore can be renamed in many ways. This is the foundation of the density principle – which is reported as a major shortcoming in upper primary and secondary students' competency with fractions (see Braithwaite & Siegler, 2018; Jordan et al., 2016; Siegler, 2013; Thomson, 2020; Ubah & Bansilal, 2018). The present study demonstrates that explicit experiences with physically comparing the spatial and geometrical attributes of the fraction parts created, in addition to visualising the spatial and geometric structures and proportions of these parts, enables children to make sense of the ways in which the quantity can be measured. What this demonstrates is an early generalisation about multiplicative relations of fractions because they are interpreting (reinitialising) the whole as different multiples of units (Confrey, 1994).

7.3.2.2 Recognising Proportional Equivalence

The children's understanding of the fraction as a measure meaning was also highlighted in the way they demonstrated the key indicator of recognising proportional equivalence. This key indicator focused on the unit fraction and equivalent fraction ideas in the context of comparing unlike wholes, through lessons that involved finding different fractional parts of pathways on maps (e.g., Lessons 8-10: The dinosaurs have escaped). The children were asked to compare and recreate pictorial representations that were scaled versions of the physical maps. These lessons related to the similarity and scaling ideas from the fraction as an operator meaning. As described in Chapter Three, the similarity and scaling ideas derived from Confrey et al.'s (2014) framework focus on the properties of proportional relationships between the magnitude of objects and shapes.

As children became more confident with recognising equal parts and the relationship to their size from the partitioning experiences earlier in the local instruction theory, they often referred to *halving* as a benchmark to justify the length of pathways on maps. This type of reasoning was exemplified in the whole class discussion with Class B during the mapping activities in Lessons 8–10. During this, Child 43 stated:

> …but I could imagine that I can walk from here to far, far away, and its only half to where I'm going—like Adelaide or something… I could walk from here to that table, and it's halfway of this room. You have to think about what the end is to know how big you've walked.

Similarly, Child 48 stated that the length the path was irrelevant. To be half, there simply needed to be two equal parts within the concerning path. The critical part was to first determine (interpreted as visualising) the length of the whole path. Several children explained their drawn representations of their scaled maps with statements like, 'half is going to look different on each, but still in the middle [of each path]' (Child 32).

These illustrations of thinking demonstrate that the children were conceptualising proportional equivalence (Pedersen & Bjerre, 2021). That is, they recognised that to determine a fraction of an object (a pathway in this case) needs to be in relation to the whole (fraction as a measure understanding). This reasoning was supported by the fraction as a relation meaning, specifically the distribution and equipartitioning multiple wholes ideas to compare the same fraction of two different measures (e.g., halfway on paths of different lengths). This is a very complex understanding that emerged through this study. Importantly it was spatial reasoning that enabled children to develop this level of sophistication in their thinking.

The importance of this finding extends the literature on what is known about developing young children's capacity to work with an extended range of fraction meanings. While there are several studies that demonstrate children as young as four can reason in spatial proportional contexts (e.g., matching juice and water ratios) the explicit connection to spatial proportional reasoning and children's understanding of the various meanings of fractions remains unclear. Möhring et al. (2018) reported 8 to 10-year-old children's spatial proportional reasoning capabilities were associated with their formal understanding of fractions, examined through their ability to name fractional measures presented in non-symbolic, continuous contexts (e.g., estimating the ratio of cherry juice to water mixtures presented pictorially). In reporting this finding, however, the authors questioned whether it was children's previous formal fraction instruction or their proportional reasoning competency that enabled the children to successfully complete such tasks. The present study suggests that even with little prior knowledge and experience with formal fraction instruction, spatial proportional reasoning supports the development of the fraction as a measure meaning in developing the ability to compare proportionally equivalent fractions between unlike wholes.

While this understanding was developed primarily in continuous contexts within the intervention, it provides an important foundation for children to work with more complex multiplicative ideas of fractions. That is, it demonstrates that the children could reason about the part-to-whole relationships of two separate objects (paths) whilst comparing the part-to-part relationship of the nominated fraction (Lamon, 2014). This is a key component to multiplicative thinking, and indicative of the invariant relationship that exists between proportionally equivalent fractions.

7.3.3 Fraction as a Relation

The fraction as a relation meaning introduced children to the underpinning ideas of many-to-one, distribution and proto-ratio. While only the children in Class B experienced the full range of activities related to these ideas due to the COVID-19 pandemic, the children's understanding of the fraction as a relation meaning was evident in the activities related to the key indicator of connecting multiplicative relations, and the post-intervention Task Based Interview (TBI) responses. These will be discussed in turn.

7.3.3.1 Connecting Multiplicative Relations

In the lessons exploring the fraction as a relation meaning, there was an emphasis on developing the language of 'for each' when exploring simple relationships and ratios. For example, in the lessons where children were asked to represent the number of dinosaur steps 'for each' human step, or the number of pies baked 'for each' dinosaur. The focus on encouraging children to describe their thinking using a 'for each' statement was intentional in order to support the development of the many-to-one idea. This is the basis for exploring how a fraction can represent a simple ratio.

A small group of children provided evidence of this thinking when working out a dinosaur to human step ratio: 'We did, one [dinosaur] step, three of our steps, one [dinosaur] step, three [of our] steps… you just keep these lines going to see the groups'. In this example, the children used a similar gesture that was associated with building up or repeating the structure of the many-to-one relationship indicating the understanding of the invariant property of simple ratio. Here, the child is considering three human steps as a composite unit, to coordinate the structure of the many-to-one relationship. In the context of the fraction as a relation meaning, the coordination of the composite unit of three human steps per one dinosaur step is the primitive foundations of multiplicative reasoning (Confrey & Smith, 1995)

This thinking was also reflected in the way many children used pictorial representations to illustrate this understanding, where they typically used row and column structures to organise the part-part quantities to represent the ratio ideas. Spatial structuring enabled children to represent these problems accurately and was also influential in how children not only worked with the many-to-one idea, but also connected multiplicative relations between other fraction meanings. For example, in the Feeding Dinosaurs tasks, Child 42 demonstrated a flexibility between the simple ratio and a connection to the fraction as a measure meaning, which was supported through their use of gesturing (Figure 5.31 reproduced below as Figure 7.3 for convenience).

Figure 7.3

Child 42's Representation of the 'Feeding Dinosaurs' Task

Child 42 stated they lined the pies up so they could see if they were equal, referring to the *distribution* of the pies to the dinosaurs. Then they gestured over the top of this representation how they ‘could see’ that two dinosaurs’ share (6 pies) was one-third of the total number of pies (18)—a composite unit idea that supports the fraction as a measure meaning.

The emphasis on spatial structuring enabled children to make multiplicative connections about this context (e.g., viewing the representation as a many-to-one or part–part quantity in addition to naming it as a composite unit, or part-whole quantity).

Understanding of the fraction as a relation meaning was also evident in the way the children’s utilised spatial visualisation to describe the different quantities as a proto-ratio. Child 47’s discussion with me (see Chapter Five, section 5.4.4.1.3) about their understanding of sharing 12 cookies between eight people highlights this connection.

> I can see in my head how you just cut all the cookies in half, and then I move them around to put them in groups of three [halves]—like, all lined up. I don’t even need to write it down; I just do it in my head!

During this interaction, I asked the child several prompting questions to further probe their understanding of the fraction as a relation meaning. When I asked them to describe how many cookies would be needed for four people, if they were each given 1-and-a-half cookies, they replied while gesturing their arms like a balance scale:

> That’s easy—it’s six! Because when you go up this side by that amount [double the number of children], you need to here as well [double the number of cookies]. But everyone still gets the same amount [of cookies]—1 and a half.

This is evidence of the proto-ratio idea, which Vanluydt et al. (2022) refer to as the many-to-many idea. In their study investigating the development of proportional reasoning in children from 5 – 8 years of age, they found that the many-to-one idea provided an essential steppingstone for the proto-ratio idea to develop. While the present study supports Vanluydt et al.’s (2022) findings, it also indicates how visualising and gesturing the structure of the quantities revealed the extent of their understanding. In this example, the child's ability to visualise the doubling of the quantities (supported by their gesture) is a demonstration of connecting multiplicative relations as they are articulating the relationship between how *many* times bigger the quantities are increasing by, while preserving the inverse relationship. The associated gesture suggest they have visualised a two-dimensional understanding of fraction as a relation (Confrey et al., 2014). This mans, they recognised the relationship between the two quantities needed to be preserved, even when they were 'building up or building down' the number of cookies and shares. Moreover, it demonstrated the emergence of multiplicative relations, as they were able to work 'in such a way that one of the composite units is distributed over the elements of the other composite unit' (Steffe, 1994, p.19). That is, the child demonstrated the coordination between units of equal size, the number of units per group and the total quantity.

In summary, the children's capability to work with the fraction as a relation meaning was developed though visualising and gesturing spatial structural changes when many-to-one and proto-ratio contexts were explored. This highlights the intervention not only enabled children to develop early fraction as a relation ideas through an emphasis on spatial structuring, but also children's understanding of the multiplicative relations between the fraction meanings more broadly. For example, the ability to connect multiplicative relations were supported through integrating the fraction as an operator ideas (specifically doubling/ halving, times-as-many and partitive division/ recursive multiplication) in the exploration of simple ratio (e.g., many-to-one and proto-ratio ideas). Furthermore, children's use of spatial structure enabled them to link fraction as a measure meaning to the various quantities, such as the many-as-one idea observed in the Feeding Dinosaurs activity. This finding provides salient evidence for how these fraction meanings are tightly connected, and how spatial visualisation and spatial structuring are strategies for building children's awareness and flexibility for working with such complex ideas.

7.3.3.2 Post-Intervention Task-Based Interviews

The post-intervention Task Based Interview (TBI) data also provides evidence that the children developed an understanding of the fraction as a relation meaning, influenced by spatial structuring and visualisation strategies. For instance, Item 23 (Plant growth rates) asked children to determine which plant had grown more within a year. The information provided was that Plant A had grown 5 cm in half a year and Plant B had grown 8 cm in a whole year. This was a complex problem because the children needed to recognise that the respective measures of the plants did not reflect the same time period, nonetheless 15 children correctly answered this item in the post-intervention TBI in comparison to four in the pre-intervention TBI. The children predominantly referred to chunking the periods of time (where they described ‘seeing’ – interpreted as visualising – half a year in comparison to a whole year), to compare the plant’s rate of growth. This type of explanation was often accompanied by children drawing lines to represent the many-to-one relationship between the period of time (one year) and the measurement of the plant, as described in Chapter Five (section 5.5.2.2). This evidence would suggest that the children were drawing on the doubling and halving idea from the fraction as an operator meaning, which was overtly supported by spatial visualisation and spatial structuring as discussed above. This finding demonstrates the critical link spatial reasoning plays for enabling children to connect the different meanings of fractions, again, demonstrating early multiplicative foundations of their reasoning.

The development of children’s fraction as a relation understandings were developed though visualising spatial structural changes when many-to-one and proto-ratio contexts are explored. This highlights the intervention not only enabled children to develop early fraction as a relation ideas through an emphasis on spatial structuring, but also children’s multiplicative relations between the fraction meanings more broadly. For example, the ability to connect multiplicative relations were supported through integrating the fraction as an operator idea (specifically doubling/ halving, times-as-many, and partitive division/ recursive multiplication) in the exploration of simple ratio (e.g., many-to-one and proto-ratio ideas). Furthermore, children's use of spatial structure enabled them to link fraction as a measure meaning to the various quantities, such as the many-as-one idea observes in the Feeding Dinosaurs activity. This finding provides salient evidence for how these fraction meanings are tightly connected, and how spatial visualisation and spatial structuring are strategies for building children's awareness and flexibility for working with such complex ideas.

7.3.4 Summary of Research Question One

The present study demonstrates young children can develop complex and flexible understandings of the three meanings of fractions, however it is the emphasis on spatial reasoning—namely spatial visualisation, spatial proportional reasoning, and spatial structuring—that is critical to helping children develop this understanding. For example, the children demonstrated fraction as an operator knowledge through their ability to predict the outcome of fair sharing situations through visualising the operations (e.g., doubling/ halving, partitive division/ recursive multiplication, times as many) performed on both continuous objects and discrete sets. The fraction as an operator meaning supported by spatial visualisation was evident throughout the intervention; however, this fraction meaning, and spatial construct were highly impactful in children's development of the creating and justifying equal shares key indicator.

The children's fraction as a measure understanding was evidenced by their ability to work with unit fractions, composite units, and equivalent fractions. This was observed in lessons that developed their ability to reinitialise the unit and recognise proportional equivalence as the next two key indicators of the local instruction theory. This understanding was supported by spatial structuring in discrete contexts specifically, and spatial proportional reasoning in continuous contexts. The extent to which children developed an understanding of the fraction as a measure meaning was evidenced by their appreciation of early quantitative equivalence and fraction magnitude knowledge.

Evidence of the fraction as a relation meaning was observed in the lessons related to the final key indicator – connecting multiplicative relations. The children demonstrated this by their ability to work with and describe many to one and proto-ratio ideas, when preserving the invariance relationships of simple ratios. Spatial structuring and spatial visualisation were critical constructs the children utilised in demonstrating this understanding. The children's post-intervention TBI also revealed the role spatial structuring and spatial visualisation had when working with a complex rate problem, where children drew on fraction as an operator and fraction as a measure ideas to connect different multiplicative relations (e.g., doubling and halving, partitive division, composite units, and equivalent fractions).

Also evident across the intervention, including in the TBI, was the children's use of gesture and, to a lesser extent, spatial language. The use of gesture was not included as a pedagogical approach in the intervention, since there has been little attention given to children's self-initiated use of gesture in the current literature (Krause & Salle, 2019). Yet these forms of representations provided insights into, and evidence of, the spatial reasoning constructs the children were utilising when tackling the different fraction meanings.

In summary, while Confrey et al.'s (2014) framework articulates the connections between fraction as an operator, fraction as a measure and fraction as a relation meanings, the present study provides fundamental evidence of how a spatial reasoning approach facilitates this understanding. Importantly, the emphasis on working with these meanings through a spatial reasoning approach enabled children to demonstrate an appreciation of the multiplicative nature of the three meanings of fractions, well beyond what is expected for this age group.

The next section will answer Research Question Two, which was concerned with the intervention's impact on young children's whole number knowledge.

7.4 Research Question Two

To what extent, if any, does this approach to fractions impact young children's understanding of whole number?

The TBI provided data will be used to answer this research question. In relation to children's whole number knowledge, the assessment included four subitising tasks and five place value tasks (Siemon, 2006). As outlined in Chapter Three, the purpose of including whole number questions was to gain insight into children's general number knowledge. Given that the intervention was designed to help children visualise and develop a sense of magnitude about quantity, which included fraction and whole number relations (such as simple ratio), I wanted to explore if and how children's whole number knowledge was influenced by the intervention activities. The extent to which children demonstrated an improvement in their understanding or strategy choice within whole number knowledge, was evident in two main areas: their part-part-whole understandings, and whole number magnitude knowledge, characterised by the children's improvement in their understanding of the relative size of whole numbers. Both are discussed below.

7.4.1 Part-Part-Whole Relations

The results of the post-TBI assessment of subitising indicated an overall improvement in the children's part-part-whole knowledge, specifically for numbers up to 12. It is argued that the integration of spatial structuring throughout the intervention, in discrete contexts, contributed to this improvement. For example, the children's experience with explicitly visualising and describing quantities based on their structure in the representation (e.g., considering six as two rows of three, three columns of two, etc.) enabled them to make connections in both whole number and fraction contexts. Statements like Child 61's description of how they subitised 10 presented in a tens frame—'10 is two rows of five. The fives are halves of the 10—just like in fractions'—are an example of how many children demonstrated a flexible understanding of how they saw the structure of the quantity as different composite units, and flexibly named them between fraction and whole number parts.

This thinking is further reflected by children's description of the many-as-one understanding from the fraction as a measure meaning, and how pattern and structure transferred into the relationships children developed in the whole number contexts. For example, children referred to 'seeing' various structures such as 'triangles of dots' in Item 4 (Tens Frame Bananas). Although the task asked the children to describe how many bananas there would be if three were added to the group of six presented in a tens frame, one child referred to the geometric (triangular) structures of the '3-thirds' as triangles of bananas (referring to their arrangement in the representation). Similarly, many children identified half of a set of 16 stars as eight in Item 16, by describing how they 'saw' two groups of four stars, also indicated the connection between spatial structure and children's part-part-whole number accuracy. This contrasts with the pre TBI where many children were observed counting individual stars.

The analysis of the post-intervention TBI also revealed a shift in children's level of flexibility about part-part-whole relations related to place value tasks. Child 32's explanation of what the '2' and '6' refer to in the number 26 is an example of the type of structural awareness of numerical magnitudes evident in many children's post-assessment responses: 'You can see 26 is two groups of 10 and then six [more]…but is also two groups of 13—I just saw that! An increased number of children in the post-intervention TBI, made flexible statements about the spatial structure of the different underlying units, suggesting a deep understanding of composite (whole) units. It also suggests that children had started to develop early ideas about multiplicative relationships for various small collections, because they demonstrated an understanding of different abstract composite units (Cobb, 1995) such as 26 is two tens (and six) or two thirteens.

As Cobb and Gravemeijer (2014) describe, identifying how a particular form of reasoning employed as a strategy in the intervention, and its effect on the reorganisation of other forms of reasoning, is an important component to the retrospective analysis of DBR studies. In this case, the awareness and assignment of spatial structure in the development of range of fraction, and simple ratio ideas, appears to have helped children think and work with whole number quantities in a more sophisticated manner. That being, the ability to visualise and work with different composite units with whole number quantities.

This evidence of part-part-whole thinking through exploring the *spatial structures* in representations suggests the intervention improved children's understanding that whole numbers can be decomposed into, and composed of, smaller units (Kullberg, 2020; Resnick, 1983). This is evidence of reinitialising the unit in both whole number and fraction contexts. This understanding was not achieved through counting single units, but by working with composite units of whole numbers to explore unitising, and early arithmetic skills.

The acts of dividing, sharing, reproducing copies of parts, and reassembling quantities the children experienced in the present study were influenced by the spatial structure children developed and assigned to the representations to support an understanding of the relationships between the quantities. As Battista et al. (1999) state, spatial structuring is an activity of the individual that is constructed in the child's mind; it is not 'in' the representations themselves, but a personal construction or interpretation of the representation that the child makes. It is the abstraction of the structure of the context itself that supports the children's ability to reason about the quantity. In this study, the children developed the ability to assign spatial structures to whole number and early fraction part-part relationships. This ability to work between whole number and fraction ideas – specifically with the abstraction of structure of composite units, is the basis early multiplicative thinking (Siemon & Breed, 2006). To this point, these findings demonstrate that the local instruction theory enabled children to develop a broader range of rational number ideas that reflected the structural relationship between whole number and fraction quantities through a spatial reasoning approach.

7.4.2 Whole Number Magnitude

Evidence of a significant shift in children's understanding of whole number magnitude was reflected in the responses to the post-intervention TBI. For example, in Item 8 (Proportional number line), over half of the children in the pre-intervention interview exhibited a counting based strategy to place the numbers eight and 16 on an unpartitioned 0–20 number line (with only five across both classes attempting the 0-100 number line). In the post-assessment, 18 children were able to complete both the 0–20 and 0–100 number line tasks accurately. However, there was a noticeable change in the strategies children used to tackle this task. These strategies indicated they were engaging with spatial proportional reasoning. For example:

> Child 42: You just think half of 100 instead of half of 20 to work out where the numbers go, even though the line looks the same [length].
>
> Child 44: It's like half of something can look the same [running finger along number lines], but you have to think about what the whole total is, like half is 10 [pointing to the 0–20 number line], but [half] is 50 here [pointing to 0–100 number line], but they are both the same [length].

This thinking indicates a developing sense of proportional equivalence, whereby the child is using strategies such as benchmarking to the halfway mark to reason about their whole number knowledge. While working with two number lines of the same absolute length, the children were able to recognise that the density of each number line was different. Using spatial proportional reasoning to benchmark halfway on each line, demonstrates that the children appreciated the size of the quantities required in relation to the whole, without being distracted or confused by the length of the line itself, as was seen in the pre-intervention data.

These findings again highlight the connection between mathematical and spatial structures (Hino & Kato, 2018), specifically the influence spatial proportional reasoning has had on children's whole number magnitude understandings. In comparison to the theoretical perspectives on numerical magnitude knowledge that suggest it develops from whole number foundations (e.g., Olive, 1999; Siegler et al., 2011; Steffe & Olive, 2009), the present study's findings offer insightful evidence that demonstrate the power of a spatial reasoning partitioning-based approach, can positively influence children whole number magnitude understanding.

7.4.3 Summary of Research Question Two

The results of the TBI reveal that the intervention had a substantial influence on children's part-part-whole number knowledge and whole number magnitude understanding. Due to the nature of the TBI, identifying the exact reasons for the improvement on the children's post TBI data is difficult. However, it is reasonable to suggest that the focus on spatial structures in discrete contexts to explore the creation and justification of fair shares, and the reinitialisation of fraction units supported children's subitising capabilities and their ability to view whole numbers as abstract composite units. That is, it appeared that children's capacity to recognise and name composite units and equal groups through utilising spatial structures in various representations improved their ability to subitise the part-part structures, and work with composite units in place value problems. Similarly, the improvement in children's whole number magnitude knowledge was supported by spatial proportional reasoning and the doubling and halving idea, in their ability to reason about the relative magnitude of whole numbers, when working with number lines (e.g., 0-100).

7.5 Zooming Out: Examining the Significance of the Findings

As part of the retrospective analysis phase of this DBR, this section discusses the broader significance of the findings, considering the current theoretical perspective on young children's rational number development. This section will compare the findings to current approaches for teaching fractions, to highlight the contribution this study makes to the field of early childhood mathematics education.

To discuss the study's significance, I have framed the discussion in three parts to highlight the most salient contributions it makes. The first is a discussion about introducing early fraction ideas through a partitioning and spatial reasoning approach. The second significant contribution this study makes concerns what is currently known about the role of spatial reasoning, and its positive influence on mathematical competency. The third part addresses the efficacy of the local instruction theory and the contribution this research provides for further implementation and refinement.

7.5.1 Developing Early Conceptual Understandings of Fractions

There are two distinct interpretations of partitioning: one multiplicative, – known as splitting/ equipartitioning, (e.g., Confrey & Scarano, 1995; Confrey & Maloney, 2010) and one based on unit iteration (Cortina et al., 2014; Steffe, 2010). As described in Chapter Two, the design of this study was based on Confrey's (2008) perspective of partitioning (also referred to as equipartitioning or splitting) which suggests this approach is the basis for the development of all rational number knowledge. In this context, partitioning is considered a primitive construct that develops early in young children's lives, but in parallel to whole number knowledge (Confrey, 1994). This perspective considers partitioning as multiplicative because the origin for splitting is '1'. That is, to determine a unit fraction, the whole is considered as the starting point for *n*-splits to be applied as repeated multiplication (Confrey & Harel, 1994). In contrast, Steffe's (2010) perspective of partitioning has an origin of '0', whereby a unit fraction is created from the whole and iterated to recreate the whole, as repeated addition (see Chapter Two).

Despite these differences in perspectives on partitioning, much of the literature on the early development of fractions, states partitioning is the most authentic and meaningful pathway to developing fraction and rational number ideas more broadly (e.g., Behr et al., 1983; Confrey & Maloney, 2010; Kieren, 1993; Mack, 1990; Pitkethly & Hunting, 1996; Siemon, 2003). However, Cortina et al's (2014) research draws upon Freudenthal's (1983) discussion of this approach to present several criticisms with this perspective of partitioning. They state that by starting with an origin of '1', this limits children's thinking to part-whole and proper fractions. They suggest starting with '1' implies the nth part resides exclusively within the whole, limiting the children's ability to work with fractions beyond the part-whole meaning. The evidence presented in this study, however, challenges this view as it demonstrates that young children are able to work with an extended range of fraction meanings that go well beyond the part-whole meaning. Moreover, the emphasis on spatial reasoning has enabled children to develop mental models of an extended range of fractions, which has resulted in children moving from *qualitative* compensation to *quantitative* understandings of the three meanings of fractions. This is a significant contribution to what is currently known about young children's potential for rational number reasoning because it demonstrates they have developed an early understanding of fractions as multiplicative relations.

Confrey et al., (2014) refers to *qualitative* compensation as the ability to notice the change in size of a part as a result of partitioning (e.g., as *each share becomes smaller when I share my cookie with more people*). The focus on spatial reasoning in this study enabled children to develop their qualitative awareness of an extended range of the fraction ideas, well beyond the part-whole meaning. This understanding was evident in children's descriptions of how they visualised the spatial attributes of the materials to make predictions about the outcomes of partitioning and distributing shares in association with how they are named. This is evidence of early fraction magnitude understanding which is recognising that *how* the number of sharers will either increase or decrease the size (magnitude) of the shares (Corley, 2013; Confrey, 2012). As reported in Chapter Two, fraction magnitude is critical for children to understand and work with a broad range of rational number concepts throughout their years of schooling (e.g., Bruce, 2013; Confrey et al., 2015; Jordan et al., 2017; Matthews & Ellis, 2018 Siegler et al., 2011), however this study demonstrates that a focus on spatial reasoning has enabled the multiplicative foundations of partitioning to develop this understanding of magnitude.

Although the role of partitioning in the development of fraction knowledge and confidence has long been recognised (e.g., Confrey 2008; Kieren, 1993; Siemon, 2003), the findings of this study suggest that this needs to be supported by observing and visualising the geometrical transformations of the objects the children are partitioning. What this study shows is that a focus on these spatial transformations of objects supports children to appreciate the simultaneous partitioning operation that provides a multiplicative understanding of quantity, as opposed to an additive basis implied by the literature underpinning current curriculum expectations. That is, the focus is often on counting the number of parts created, rather than examining the size of the parts generated in relation to the original whole (Gould, 2011). In the present study, it was the initial focus on visualising the act of fair sharing, doubling, and halving and reassembling objects and sets to name composite and equivalent units and ratios, which enabled children to think flexibly about the three meanings of fractions and to recognise how they are tightly connected. As reported in the literature, many children much older than the participants of this study experience difficulties in recognising the different meanings and contexts of fractions, and the ability to work meaningfully and efficiently with them (Callingham & Siemon, 2021; Siemon, 2016; Thomson et al., 2020). This study shows that children as young as 6-and-7-years of age are capable of this reasoning, meaning this approach provides a powerful foundation for rational number reasoning.

An important goal of the development of partitioning according to Confrey (2012) is to move from *qualitative compensation* to *quantitative compensation*. *Quantitative compensation* is recognising 'by how much' a share changes when more or less shares are created, which is the result of repeated splits and reassembly. Whilst Confrey et al., (2014b) provide examples of children in first and second grade demonstrating aspects of quantitative compensation, they emphasise that this proficiency level (and cognitive behaviours described as essential for rational number reasoning as described in Chapter Two) are cumulative, rather than strictly hierarchical. Therefore, consistent patterns of reasoning through a range of different problems are required to make sense of the children's proficiency and capabilities. In this study, examples of this thinking include children's ability to justify the outcome of creating equal shares, where they described *how much* a share had increased or decreased because of changes in the number of sharers (e.g., double the number of sharers results in parts that are twice as small). Further, spatial visualisation and spatial structuring enabled the children to identify the quantitative relationships that reflected the different meanings of fractions (e.g., ratio of pies to dinosaurs; composite and equivalent units in the Tablecloths and Pattern Block lessons). Moreover, the integration of spatial proportional reasoning with a range of representations (e.g., pattern blocks, fraction kits, maps) also enabled children to develop clear understanding about proportional equivalence, when comparing part-part to part-whole relations.

The evidence of children's quantitative compensation has implications for theoretical perspectives on the development of fraction magnitude understandings. In light of theoretical perspectives such as the Ratio Processing System (RPS) and ITND that discuss the development of magnitude (see Carraher, 1993; Lewis et al., 2016; Matthews & Ziols, 2019; Siegler et al., 2011), the present study also provides significant insights that help bridge theory and practice. The RPS and ITND suggests that humans are born with a sensitivity to 'perceive' various non-symbolic ratio representations, 'based on the relative magnitudes of components *a* and *b* rather than by the magnitudes of either component considered in isolation' (Matthews & Ziols, 2019, p. 2016). Interestingly, Gabriel et al., (2013) point out the juxtaposition between such theories that assume the sensitivity toward non symbolic ratio is evolutionally determined, and the persistent difficulty children experience with fractions throughout schooling and beyond. Bruce et al., (2015a) note that while there are several explanations (although conflicting) that account for these assumptions about magnitude processing from the field of neuroscience, they note there is still little practical applicability for how these perspective translate into the classroom. The present study presents a bridge between the theoretical and practical application by suggesting the explicit focus on spatial reasoning in these early years of fraction instruction enables children to build on their initial awareness and sensitivity to fraction and ratio magnitude, to develop flexibly understandings of the multiple meanings of fractions.

In summary, this study contributes to what is known about how children develop early fraction ideas and what is known about the current approaches to partitioning. It confirms Confrey's (2008) splitting conjecture, whereby the development of the three meanings of fractions is dependent on the foundation of fair sharing (i.e., the equipartitioning learning trajectory). Critical to this finding, however, is that it is an emphasis on spatial reasoning provides the means for the multiplicative roots of fractions to be established.

7.5.2 Advancing the Spatial Reasoning – Mathematics Connection

As described in Chapter Two, there is no question that spatial reasoning and general mathematical achievement are highly correlated (Mix & Cheng, 2012). While the role of spatial reasoning in building part-part-whole knowledge for the numbers to ten has been recognised (e.g., Bobis 2008; Clements 1999), its role in relation to the development of early rational number knowledge has not been explored. This study provides innovative perspectives about the positive influence a spatial reasoning approach can have on young children's early fraction and whole number knowledge, particularly the role gesture played in interpreting the way children were using various spatial reasoning constructs.

A significant finding of this study was the way in which children represented their spatial reasoning when communicating their ideas about the process of partitioning and fraction magnitude. Gesture was not a construct that was explicitly employed as a pedagogical tool by me or the classroom teachers during the intervention study, however, the increased use of gesture between the pre- and post-TBI and those captured during the daily lessons, provided a valuable lens for interpreting how children were developing spatial reasoning constructs in relation to fraction magnitude. This is an important finding since, as Krause and Salle (2019) describe, very little is known about the self-initiated way learners use gestures when learning new concepts. Specifically, in the context of young children and fractions.

In relation to partitioning, a third of all gestures captured across the entire intervention were classified as iconic, those being associated with 'chopping, sawing, or cutting action' in relation to how the child was describing the act of partitioning at the time. Analysing how the children were using these gestures, suggested they were connecting the act of creating equal shares with how they gestured radial cuts on a circle, or cutting repeatedly in half a square or rectangle.

The second most common form of iconic gesture captured across the intervention, was described as a ‘cupping’ or ‘dealing’ gesture. These cupping gestures were associated with many-as-one and composite unit ideas when children were describing the size of the unit they were referring to or creating at the time, and they would cup their hands like they were holding such part or group. In addition, the ‘dealing’ gestures were associated with children’s fair share, partitive division ideas from fraction as operator, in addition to many-to-one, fraction as a relation idea, specifically when working with discrete collections. That is, while children may have physically dealt out the fair shares or parts of a physical model, when they were unable to manipulate an image, or they were required to explain their thinking, they would use these dealing gestures in the air. This suggested they were visualising the process of making groups, and an understanding of how big these group were, while referring to the recreation of the whole. An example of this was when children used both hands to gesture partitive division of cookies into four groups but referred to the four groups as a ‘square’ (each corner representing a part of the set) that recreated the whole. These are emergent understandings and were not always associated with a description (e.g., 1-quarter of 12 is three cookies, four shares of three cookies are 12). Nonetheless, the gestures indicated an emergent understanding of the relationship between the act of partitioning, the creation of units and the whole, based on their understanding of the spatial and geometrical structures of the physical objects they were examining and/or visualising.

Other iconic gestures recorded in the intervention were classified in the form of hands moving together or apart, indicating a shrinking or enlarging action, and children outlining a geometric figure. The latter gesture was typically motioning the shape of a part in relation to a whole when describing the spatial dimensions (size) of specific parts or groups, and geometric attributes of the various shapes and spaces explored. Gestures are said to provide evidence of the mental images and transformations a learner may engage with, while solving tasks that are spatial in nature (Alabali, 2005; Hostetter & Alibali, 2008; Patahuddin et al., 2018). Other studies suggest that children's gestures helped support them when thinking about, and working through, a range of spatial reasoning tasks (Logan et al., 2014; Ng & Sinclair, 2013). The present study aligns with these findings, suggesting that the children were using gesture as a support mechanism for the thinking about the spatial contexts in which fractions were explored. The increase in gesture over the intervention could be attributed to the children's confidence and ability to visualise different fraction parts and transformations as they participated in the intervention. Alternatively, it also may indicate they had not fully established a working understanding of the ideas, as Logan et al., (2014) found that children's use of gesture slowly declined as their conceptual understandings and confidence became more robust.

The children in the present study gestured both dynamic and static forms of information. That is, the children would freely gesture the static attributes of shapes and arrangements of various fraction models (e.g., the outline of a shape or the arrangement of a set of objects). They also gestured 'movement' operations such as how they visualised splitting, reassembling, increasing, or decreasing various objects and sets. Ng and Sinclair (2013) suggest the combination of dynamic and static gestures as an important finding, because it provides further insights for how gestures communicate conceptual and procedural understanding of mathematics. Furthermore, the gestures provided observable details of the strategies children use to communicate and defend the connections between the cognitive and conceptual mechanisms of the mathematical context (Alabali, 2005). Given there is scarce literature that examines children's spontaneous use of gestures in the learning of mathematics, its emergence in this case with young children and in relation to fractions demonstrates that this is an important and powerful means of identifying and supporting children's mathematical reasoning.

7.5.3 Applicability of the Local Instruction Theory

The goal of this study was to explore the extent to which children could develop an understating of the three meanings of fractions and in what ways spatial reasoning supported this learning. Examining the creation and refinement of the local instruction theory by exploring the key indicators in relation to the children's actual process of learning enables the applicability of the local theory to be determined, and provides insights for more encompassing theories (Cobb, 2003).

An important consideration of the development of a local instruction theory is that the envisioned sequence of learning goals – referred to in this thesis as key indicators, should not be interpreted as strictly hierarchical. Rather, they should be viewed as a coordinated and interrelated pathway for developing teaching programs that provide the most beneficial and authentic sequencing of ideas and opportunities to explore concepts (Gravemeijer & Van Eerde, 2009). While the starting point of the conjectured theory was an explicit focus on creating and justifying equal shares, the children had the opportunity to continue to conceptualise this key indicator throughout the intervention program. That is, in the lessons that focussed on the key indicators of reinitialising the unit and recognising proportional equivalence, the children demonstrated a flexibility and extended understanding of what equality means, in the given context (e.g., recognising that a set can be partitioned and described by different equals units; recognising that comparing "halves" of different objects requires a proportional understanding of equality). Similarly, equality was also explored and required in the last key indicator, where children developed an appreciation for the invariant property underpinning the units of a ratio. What this demonstrates is that the local instruction theory enabled mathematical generalisations of equality, to evolve throughout the coordination of the ideas within the local instruction theory, the strategies (e.g., spatial reasoning) and mathematical practices developed (gesture, language, and representations) because of the intervention.

The second aspect of exploring the applicability of the local instruction theory is that it serves as a framework for developing learning programs (e.g., hypothetical learning trajectories) to teach a specified area of mathematics (Gravemeijer, 2012). The lessons and activities designed in this intervention were designed for children to develop and achieve each key indicator. Yet, as Gravemeijer (2004) states, local instruction theories are not the explication of fixed teaching sequences. Instead, they should offer a bridge between innovative theoretical assumptions about the types of learning possible and the ways in which that learning is supported, and classroom practice. This was evidenced in Chapter Four, where the tasks and learning sequence were refined because of the children's responses and potential for learning. In addition, the tasks and sequence were refined in Chapter Six because the needs of Class C appeared to be quite different to that of Class B. As described in the teaching experiment for Class C, the children required a more sustained focus on the development of the first key indicator (creating and justifying equal shares) as they demonstrated a general lack of understanding about what constituted equality – particularly in continuous contexts. Despite the actual learning sequences being different in the teaching experiment phase, the local instruction theory enabled children to explore and develop a range of early multiplicative foundations to rational number knowledge as a direct result of participating in this intervention (Gravemeijer and Cobb, 2013).

In summary, the local instruction theory for developing and extended range of fraction meanings though a spatial reasoning approach has contributed significant insights into what is currently known about young children's potential for this area of mathematics. As Cobb and Gravemeijer (2008) state, the intent of undertaking DBR (and constructing local instruction theories) is not ‘just to develop more effective instructional approaches for addressing traditional instructional goals, but to also influence what the goals could be by demonstrating what is possible for students’ mathematical learning’ (p. 69). This study clearly demonstrates that the overarching goal for teaching and learning fractions in the early years, should be based on a multiplicative (partitioning) foundation for fraction meanings supported by spatial reasoning. This goal is justified through the findings presented in this study that demonstrate the depth of understanding possible, which mitigates many of the persistent difficulties children experience, as reported in the literature. Furthermore, local instruction theories should be explored through additional design-based studies, to gain additional insights into what types of learning are possible, and how such learning can inform a domain-specific theory (Prediger et al., 2015). The present study provides a substantial justification for the exploration and development of a domain-specific theory that explores the power of spatial reasoning, in early rational number reasoning.

7.6 Chapter Summary

This chapter has provided a discussion on the findings of this study, as part of the retrospective analysis phase. The study demonstrated that young children developed complex and sophisticated understandings of fraction as operator, fraction as a measure and fraction as a relation meanings, through a spatial reasoning approach. The discussion of the first research question highlighted how the children evidenced their understandings of core concepts that underpin these meanings, namely partitioning, unitising and quantitative equivalence, which indicated early multiplicative understandings of fractions. Spatial visualisation is a key construct that supports the partitioning concept, and underpins the fraction as an operator ideas, primarily because it enabled children to move from physically creating shares to predicting and visualising the outcome of operating on various quantities. The emphasis on creating spatial structure patterns and arrangements in representations that reflected much of the geometrical and physical transformations children were visualising, enabled children to develop fraction as a measure ideas to name and quantify various parts and relationship between parts. This understanding reflects a multiplicative appreciation of the quantity, as a whole being the product of the parts created (Vergnaud, 1988). This emphasis on structure and arrangement also enabled children to develop understanding of simple ratio. Moreover, spatial proportional reasoning was influential in how children developed their understanding of equivalent relationships between like and unlike wholes.

The children's use of gesture and spatial language throughout the study provided additional ways to interpret how they were thinking and working with the fraction ideas, revealing ways to support both conceptual and procedural understanding. Much of the children's gestures supported their explanations of how they were visualising the size and structure of parts, and how they predicted the outcome of creating different sharing situations.

The impact this study had on young children's whole number knowledge became the basis for Research Question Two. The children demonstrated an improvement in part-part whole relation and whole number magnitude. These improvements were attributed to the focus on the spatial structures and flexibility between naming and reinitialising various units in relation to a whole. The local instruction theory has provided a framework for how the fraction as measure, fraction as an operator and fraction as a relation meanings can be authentically introduced and developed in the early years of primary school, and showcased that spatial reasoning plays a powerful role in this development.

Hawes et al. (2015) state that mathematics is an inherently spatial activity that connects to many learning areas within this discipline. This study provides more evidence of the spatial-mathematical connection in a domain not typically considered 'spatial' such as fractions. This research advances the field with practical, significant findings that provide new theoretical insights into what is currently known about the connection between early fraction understandings and spatial reasoning and how it supports early multiplicative foundations to rational number reasoning. Additionally, it provides substantial insights into young children's capabilities that can be used to inform current policy and curriculum standards. The implications for practice, limitations of the study and recommendations for future research will be explored in the next chapter.

Chapter 8: Conclusion

8.1 Chapter Overview

This chapter begins with a summary of the key findings from the study in section 8.2. This will reiterate the types of fraction understandings young children are capable of developing through a spatial reasoning approach and highlight the early multiplicative ideas the children developed as a result of this intervention. The implications and limitations of the study related to pedagogical and curricula considerations are presented (sections 8.3 and 8.4), and recommendations for further research are discussed in section 8.5. Finally, I share my final reflections of the study to conclude this thesis in section 8.6.

8.2 Summary of the Findings

This study investigated the extent to which young children could develop an understanding of the fraction as operator, fraction as a measure and fraction as a relation meanings, though a spatial reasoning approach. The connection between spatial reasoning and an extended range of early fraction understanding has not previously been explored in research literature. A series of teaching experiments were conducted to explore a local instruction theory that described a conjectured sequence of fraction understandings and how spatial reasoning supported this learning. The findings demonstrate that young children can develop complex and flexible understandings of the three meanings of fractions that indicated early multiplicative understandings of fractions and whole number properties. However, it is the emphasis on spatial reasoning—specifically, spatial visualisation, spatial proportional reasoning, and spatial structuring—that were critical for supporting this learning. The local instruction theory is re-presented in Table 8.1.

Table 8.1

The Local Instruction Theory (Version Three)

Key Indicators	Characteristics of Tasks	
	Primary Fraction Foci	**Spatial Reasoning Approach**
Creating and justifying equal shares	**Fraction as an Operator:** Fair shares Doubling/ halving Partitive division/ recursive multiplication, Geometric symmetries, Similarity **Fraction as a Measure:** Many as one, Measure, Composite units, Unit fraction	Visual perception of equal groups (drawing on spatial structures and arrangements). Equality of parts regardless of model (i.e., equal parts for discrete collections and continuous models less than and greater than 1). Visual awareness of the geometric properties of parts and sets (e.g., shape, orientation, pattern, symmetry). Observing the physical transformations of partitioning (dividing and reassembling), and visualising and predicting the outcome of a nominated split (e.g., spatial visualisation)
Reinitialising the unit	**Fraction as a Measure:** Composite units, Unit fractions, Part-whole fractions, Equivalent fractions **Fraction as an Operator:** Fair shares, Doubling /halving; Partitive division/ recursive multiplication, Times-as-many, Similarity **Fraction as a Relation** Many-to-one Distribution	Visualising measures between parts and wholes, and between composite and unit fractions through unitising. Exploring the spatial structure and arrangement of objects and sets to create and compare different unit fractions. Visualising magnitude relations between parts (double/ half/ times as many) the distribution of parts to determine equivalence.
Recognising proportional equivalence	**Fraction as a Relation** Distribution, Proto-ratio, Equipartitioning multiple wholes,	Visualising the relationship between equivalent measures, of same and different wholes.

	Fraction as an Operator Doubling/ halving, Times-as-many, 1-nth-of..., Scaling, Geometric symmetries, Similarity **Fraction as a Measure** Composite units Unit fractions Equivalent fractions	
Connecting multiplicative relations	**Fraction as a Relation** Many-to-one, Distribution, Proto-ratio **Fraction as an Operator** Doubling /halving Partitive division/ recursive multiplication Times-as-many, 1-nth-of..., Scaling **Fraction as a Measure** Composite units Part-whole fractions Equivalent fractions	Early relational understandings between the structure of part-part and part-whole quantities. (e.g., visualising and justifying the relational magnitude of fractions in relation to other fractions (e.g., 1-quarter is a half of a half/ twice as small); and working flexibly with non-symbolic simple ratios (e.g., 1:2 = 2:4).

The children demonstrated fraction as an operator knowledge through their ability to predict the outcome of fair sharing situations through visualising various operations (e.g., doubling/ halving, partitive division/ recursive multiplication, times as many). The children were able to predict and visualise these operations in both continuous and discrete contexts to develop an understanding of the relationship between the number of parts and their size. The fraction as an operator meaning supported by spatial visualisation was evident throughout the intervention; however, it was highly efficacious in relation to the key indicator of *creating and justifying equal shares.*

The children's fraction as a measure understanding was tightly connected to the development of unit fractions, composite units, and equivalent fractions ideas. In discrete contexts, spatial structuring played a vital role in how children were able to view unit fractions and composite unit arrangements, to demonstrate an early appreciation of the quantitate equivalence concept. By developing this understanding of the fraction as a measure meaning, the children became competent with naming and renaming fraction relationships, demonstrating the key indicator of *reinitialising the unit*. Within continuous contexts, the children developed the key indicator of *recognising proportional equivalence,* through their ability to compare unit fractions, composite units, and equivalent fractions between different wholes. This understanding was explicitly supported by spatial proportional reasoning. The extent to which children developed an understanding of the fraction as a measure meaning suggested the early appreciation for fraction magnitude. That is, the children demonstrated early awareness of the density principle of fractions, as they were able to experiment with various structures or proportions of objects to make sense of how fractions are named and related.

The fraction as a relation meaning was evidenced by children's ability to work with and describe the many-to-one and proto-ratio ideas when preserving the invariance relationship of simple ratios. Spatial structuring and spatial visualisation were important spatial constructs the children utilised in demonstrating this understanding, which was reflected in the key indicator of *connecting multiplicative relations*. Examples of this thinking included descriptions of how the children organised and arranged the part-part structures to describe the many-to-one idea, and how they visualised doubling or halving the units to make sense of the proto-ratio idea. The children's post-intervention Task Based Interview (TBI) also revealed the influence spatial structuring and spatial visualisation played in working with a complex rate problem, where children drew on fraction as an operator and fraction as a measure ideas to connect different multiplicative relations. Finally, the children's whole number knowledge was improved between the pre- and post-intervention TBI, specifically, the way many of the children viewed whole numbers as composite units. In addition, many children demonstrating a greater appreciation for the relative magnitude of whole numbers in number line tasks. These findings suggested that the emphasis on spatial structure assisted children's understanding of whole number composite units, in addition to spatial proportional reasoning when reasoning about the density of two different number lines. This study has demonstrated that the children have developed early multiplicative foundations for working with initial rational number ideas, by their ability to recognise the relative magnitude of fractions and whole numbers. This was evidenced by the children's ability to work with double and half, times-as-many ideas; and their flexibly with partitive division/ recursive multiplication, composite units and the distribution of units. These types of ideas and skills were developed as a direct result of the local instruction theory, as they are not supported by the current curriculum requirements for children at this age.

Another important finding across the intervention, including in the TBI was the children's use of gesture and, to a lesser extent, spatial language. These forms of representations provided insights into, and evidence of the spatial reasoning constructs the children were utilising when working with the different fraction meanings. The use of gesture was not included as a pedagogical approach in the intervention, because there is little known about the role spontaneous gestures (rather than teacher lead gestures) may play in the early development of fraction ideas. However, it was clear that the children used gestures to effectively communicate both spatial and mathematical knowledge, which is an essential contribution to the literature that this study provides. The implications of the findings of the study will now be discussed.

8.3 Implications

This research has shown that using spatial reasoning to explore fraction meanings positively influences young children's ability to understand seemingly complex ideas. This finding has important implications for both curriculum expectations in Australia, and the conceptual and pedagogical knowledge of teachers.

8.3.1 Curricula Implications

The way teachers implement mathematics education programs is directly influenced by the demands of the curricula standards in Australia. For this reason, it is critical that the curriculum standards reflect current research. When the curriculum is research-informed, this supports teachers to enact best-practice teaching methods. Utilising spatial reasoning constructs to support children in exploring an extended range of fraction meanings should be reflected in the curriculum. As described in Chapter One, the curriculum descriptors in the Australian Curriculum: Mathematics document (ACARA, 2023) reflect a limited range of ideas. Since the data collection period of this study, the Australian Curriculum: Mathematics document has undergone a significant update. The newest version, known as version 9.0, is currently being implemented across most states and territories. Yet, like the previous versions of the Australian Curriculum, version 9.0 does not reflect the capabilities of the children observed in this study.

8.3.1.1 Content Descriptors

When we look closely at the curriculum, we can see that in Foundation (the first year of school in Australia) 5- and 6-year-old children are required to:

> AC9MFN06: represent practical situations that involve equal sharing and grouping with physical and virtual materials and use counting or subitising strategies (ACARA, 2023).

While mentioning subitising, this content descriptor also emphasises a counting based approach to explore fair sharing. This is reiterated in the elaborations (which provide further advice for teachers on the content descriptors) which describe discrete contexts only for equal sharing and emphasise counting as a strategy to check for equality. For example,

> …sharing pieces of fruit or a bunch of grapes between 4 people and discussing how you would know they have been shared equally; when playing card games where each player is dealt the same number of cards and counting the number of cards after the deal to ensure they have the same amount (ACARA, 2023)

For teachers who lack a strong understanding of the best ways to teach rational numbers, this descriptor may encourage them to focus on the problematic double-counting idea to think about and name fractions, described as in Chapter Two (See Gould, 2011; Lamon 2007).

For these descriptors to be truly reflective of current research, it is important they include the introduction of the fraction as an operator meaning, that is underpinned by the fair sharing idea. At this age, a focus on spatial reasoning is important. An alternative description for this year level should involve creating and justifying equal shares in both discrete and continuous contexts. In addition, there should be an emphasis on predicting, visualising, and describing (using gesture and spatial language) the process and outcome of creating different fair shares. With the inclusion of visualisation in the content descriptor, the elaboration should also suggest a focus on observing and creating different patterns and structures with discrete and continuous materials, to justify the equal shares. This acknowledges the role spatial structuring plays in young children's awareness of equal shares.

While the children in the present study were older than children who would typically be in the Foundation year of schooling, establishing the idea of equality in a range of contexts is critical. Supporting this understanding in the first year of school through an emphasis on spatial reasoning strategies, is likely to provide children with a rich foundation for developing more complex ideas for the three meanings of fractions in Year 1 and 2.

In Year 1, the Australian Curriculum version 9.0 content description related to fraction states:

> AC9M1N06: use mathematical modelling to solve practical problems involving equal sharing and grouping; represent the situations with diagrams, physical and virtual materials, and use calculation strategies to solve the problem (ACARA, 2023).

The inclusion of various representations (diagrams, physical and virtual materials) is pleasing to see when working with equal sharing and grouping ideas. Yet, there is no mention of the role spatial reasoning plays when working with such models, in the context of fractions. An alternative content descriptor should include visualising, naming, and renaming fair shares in discrete and continuous contexts, by exploring and communicating how many ways the same set or object can be shared fairly. This suggested descriptor aligns with a major finding of the present study, which is that Year 1 and 2 is a critical time for supporting children to shift their reasoning from a focus on creating and justifying sharing sets or continuous models to an appreciation that the quantities can be named in multiple ways. Within the accompanying elaboration, teachers should be guided to explore the fraction as an operator ideas of doubling and halving, partitive division/ recursive multiplication, and times-as-many ideas to support visualising and communicating (e.g., gesture and spatial language) the outcome of fair sharing. Further to this, introducing children to explorations of how fractions are named and renamed based on fair sharing contexts connects to the fraction as a measure ideas of composite fractions, unit fractions, and equivalent fractions. These ideas need to be supported through comparing parts in relation to one another through spatial proportional reasoning (e.g., using pattern blocks to explore the relationships between composite and equivalent fractions). In discrete contexts, children should explore making patterns with objects, (e.g., row, and column structures) to help them consider how a whole can be named by various shares. In addition, the focus on creating structures to represent the early fraction as a relation understanding should be introduced by exploring problems such as: *who gets a bigger share of cookies: four children sharing 12 cookies, or three children sharing six cookies fairly?*

In Year 2, children are expected to develop the following skills and understanding in relation to fractions.

> AC9M2N03: recognise and describe one-half as one of 2 equal parts of a whole and connect halves, quarters, and eighths through repeated halving (ACARA, 2023)

This expectation is very concerning as it explicitly emphasises the part-whole construct and does not build on from the Year 1 content expectations. As discussed in Chapter Two, there is a wealth of literature that describes how and why focusing solely on the part-whole meaning is limiting (see Confrey et al., 2014b; Gould, 2011; Kieren, 1995; Lamon 2006; Siemon 2006). The findings of this study demonstrate that children at this age are far more capable of engaging with a range of fraction meanings that implied in this descriptor. A more suitable inclusion for the curriculum at this age would be a focus on developing children’s proportional understanding of fraction and whole number relations. This suggested content builds upon the suggestions made for the previous two years of schooling much more authentically than the current sequence. The elaboration should describe using spatial proportional reasoning to justify the similarities and differences between fair shares of different wholes, (e.g., comparing 1-quarter of the way along two different length pathways). In addition, children should also be supported to use spatial proportional reasoning for fraction as a relation problems, such as: *Would you get a bigger, smaller, or same size share if there were two chocolate bars shared between four children, or one chocolate bars between two children?* Problems like these can also enable children to explore how many ‘times as big’ one share is in relation to another (fraction as operator), again by focusing on spatial proportional reasoning. In discrete contexts, children should be focusing on spatial structures to explore ‘building up and building down’ strategies (e.g., the ‘Dinosaur Steps’ problem) to explore simple ratio and early multiplicative relations further.

It is noteworthy that a specific reference to ‘fractions’ only appears in the curriculum descriptors from Year 2 (in Year Foundation and 1, reference is made to ‘sharing’). Many teachers may not necessarily make the connection between fair sharing and early fraction reasoning. This perpetuates the assumption that children younger than Year 2 are not ‘ready’ for exploring a range of fraction ideas.

8.3.1.2 Spatial Reasoning and the Australian Curriculum

Spatial reasoning was one element of the ‘numeracy’ general capability within the Australian Curriculum (version 8.4) (ACARA, n.d). General capabilities are described as the ‘knowledge, skills behaviours and dispositions to live and work successfully’ (ACARA, 2023). They are an acknowledgement that learning, in general, requires a range of disciplinary (e.g., literacy, numeracy) and non-disciplinary skills and understandings (e.g., ethical understanding, critical and creative thinking etc.). Teachers are required to consider how they can provide opportunities for children to develop the general capabilities across all areas of the curriculum.

In the updated curriculum, spatial reasoning has been removed from the numeracy general capability. Omitting spatial reasoning from the numeracy general capability does not reflect the vast research on the connection between spatial reasoning and mathematics more generally, as discussed in Chapter Two. Not only should spatial reasoning be ‘visible’, like the suggestions made for the fraction content described above; it should be a part of the general capabilities. Including spatial reasoning into this area of the curriculum will reflect the contribution this cognitive behaviour makes to many areas of learning – specifically mathematics.

In summary, the findings of the present study suggests that the current curriculum demands for rational number knowledge are inadequate for children in the early years of school. They fail to consider the capabilities of these children; fail to promote best practice in teaching fractions and fail to highlight the importance of spatial reasoning and its connection to mathematics. The suggested content based on the result of this study reflect a much more coherent, appropriate, and evidence-informed sequence of learning for children in the early years of school.

8.3.2 Conceptual and Pedagogical Content Knowledge Implications

The second implication from the findings of this study suggest that teachers conceptual and pedagogical content knowledge needs to be supported in relation to the teaching and learning of fractions. Hurrell (2013) found that many Australian primary school teachers hold negative views about the teaching of fractions, which stems from their own lack of conceptual understanding and confidence in this area of mathematics. The approach taken in this study requires teachers to have a deep understanding of the three meanings of fractions and how spatial reasoning supports this learning. Therefore, it is imperative that teachers are supported to develop their conceptual knowledge about the three meaning of fractions and their underpinning ideas, so that the negative views are not reinforced. This can be achieved through quality professional development and pre-service teacher education programs.

The findings of this study also imply that pre- and in-service teachers need to understand the role of spatial reasoning and representations in their pedagogies for teaching fractions. For example, gesture and spatial language were important representations the children used to communicate spatial and mathematical information. As such, teachers need to be supported to recognise and understand what the use of gestures and spatial language might mean in the context of learning fractions and how they can utilise these representations to further support learning. As described in Chapter Two, there is unquestioned connection between learner's spatial reasoning performance and improved mathematical achievement (Mix & Cheng, 2012). Chamberlain (2020) discusses this point in relation to the Australian Institute for Teaching and School Leadership (AITSL) standards. She suggests that teachers may interpret the professional knowledge standards of *know students and how they learn*, and *know the content and how to teach it*, as a 'focus on abstract mental operations, which often reduce, and occasionally dismiss, the importance of embodied interactions' (p. 54). This includes a lack of awareness about the role gesture plays in the development of spatial and mathematical information.

In summary, there is increasing evidence of how spatial reasoning is tightly connected to a range of early mathematical topics (e.g., Bruce et al., 2015b; Gunderson et al., 2012; Lowrie et al., 2018; Mulligan et al., 2020; Seah & Horne, 2020), with Gilligan-Lee et al., (2022) suggesting it is a missing component to the mathematics curricula and our pedagogical approaches. The importance of pre-service and in service teachers exploring the role spatial reasoning and representations such as gesture and spatial language play in early fraction development is clear.

8.4 Limitations

There were two main limitations identified in this study. The limitations were in relation to the data collection tools and level of exposure the children received in the intervention.

8.4.1 Data Collection Methods and Tools

The observation tools employed throughout the intervention, in addition to the Task Based Interview (TBI) are the limitations associated with the data collection methods and tools.

8.4.1.1 Observation Tools

As described in Chapter Three, the sources of data collected (reflections on children's behaviour, observations of gestures and records of discussions) were all interpreted by the classroom teacher and researcher as they happened, or later from the work samples and representations collected during each lesson. This type of data collection and interpretation presents an issue of credibility. The absence of video recordings meant the context and interactions were not able to be observed multiple times. To address this issue, I regularly cross checked my interpretations with members of my supervisory team and constantly interrogated and analysed my notes, work samples I gathered, and the notes of the classroom teachers. This limitation became particularly evident when exploring the use and prevalence of gesture in the children's interactions. Without video and audio data I was unable to accurately capture the incidental gestures and the use of language by children who were working independently in each lesson.

8.4.1.2 Task-Based Interview

The Pre and Post Task Based Interviews (TBI) provided useful data to examine the impact of the intervention on three main areas: whole number, fraction, and spatial reasoning capabilities. However, the number of items in the TBI in each of the three areas limited the conclusions that could be made. Less whole number items were used compared to the fraction and spatial reasoning items. This limited the insights gathered into children's whole number knowledge. Future iterations of this research may include using other whole number testing measures, such as the Progressive Achievement Test in Mathematics (PAT-M), which is commonly administered in Australian schools. While this test is only administered once a year, it may provide additional insights into the influence a spatialised approach to teaching fractions has on children's whole number knowledge and general mathematical achievement.

In addition, when assessing the student's fractional understanding, there were a different number of items addressing each of three fraction meanings and the relevant spatial constructs. Some of these items were also created or adapted for this TBI, meaning the items had not been validated. As described in Chapters Five and Six, the statistical inferences made in this study related to children's pre and post intervention TBI success should be taken with caution. In order to address this limitation for future research, the development and validation of assessment items for fractions and spatial reasoning specifically for the early years is an important first step to determine the impact of this instruction.

8.4.2 Participant Exposure to the Intervention

The dependability of the results in this study is influenced by the sample size and timing of this study. Class A participated in the trial of the tasks designed to explore the conjectured key indicators for the initial local instruction theory. Useful insights were gained about how children engaged with the tasks in the intervention, including their use of spatial skills and representational tools (such as gesture) which resulted in additional themes being added to the codebook. However, as this cohort did not experience the intervention in its entirety, the findings of this study are primarily based on the cohort of 44 children who participated in Phase Two (Class B and C). This number (n=44) is relatively modest and needs to be considered when making theoretical and pedagogical generalisations related to the data. This was further complicated by the COVID-19 global pandemic, where Class C's intervention was limited to just five of the 13 intended lessons due to government and university restrictions. Furthermore, post-intervention assessment data was collected from only 15 of the 21 children in this cycle, meaning the statistical inferences made were from a smaller sample size. To mitigate this limitation, additional cycles of this intervention are recommended be undertaken to compare with the current findings.

8.5 Recommendations for Future Research

Three clear lines of inquiry are recommended for further research into the connection between an extended range of fraction meanings and spatial reasoning: (1) the expansion of the intervention materials for the local instruction theory across the first three years of primary school; (2) examination of how the current approach may support the transition and understanding of symbolic representations of an extended range of fraction meanings, and (3) conducting a video study to investigate children's spontaneous gestures throughout the intervention program.

8.5.1 Expanding the Local Instruction Theory

To build upon the present study, the next step is to examine the outcomes of the local instruction theory on a larger scale. For example, implementing the first key indicator of the local instruction theory into the first year of school would be advantageous to examine the effects this approach may have on children's whole number and fraction understandings.

In addition, the development of an extended range of teaching materials beyond the 13 lessons in the present study, may reveal additional spatial reasoning constructs that could be beneficial for children. That is, while spatial visualization, spatial proportional reasoning, and spatial structuring were the predominant spatial constructs explored within the current intervention program, experimenting with different ways to introduce and explore the three meanings of fractions may also reveal new spatial constructs that could be equally as beneficial in instruction. It is hoped that this study provides the impetus for other researchers to explore additional spatial constructs in relation to fractions and rational number reasoning more broadly, so that teachers (and researchers) can explore the potential of spatial reasoning in relation to this important area of mathematics.

8.5.2 The Introduction of Symbolic Notation

A second recommendation for further research would be to examine what impact this approach has on children's understanding of symbolic notation for fraction understanding. As described in the literature review, one of the main difficulties children in the upper primary and secondary years of schooling exhibit with fractions is the misinterpretation of symbolic fraction representations. The misconceptions associated with symbolic representations of fractions is closely associated with what Ni and Zhou (2005) termed, the Whole Number Bias (WND). As mentioned in Chapter Two, the bias is based on a lack of appreciation of fraction and whole number magnitude, where the fraction symbols can be misinterpreted as cardinal numbers. Kalra et al., (2020) suggest the WNB is better reframed as 'the failure to think relationally in a general sense, rather than failures of processes specific to numerical cognition' (p. 2). They also found in their study that Year 2 and Year 5 children's relational reasoning of various fraction meanings in non-symbolic (pictorial) contexts, was directly associated with children's interpretation of symbolic representations of fractions. However, this study acknowledged that the mechanism for developing young children's relational reasoning in the context of fractions has not been widely explored in the primary years of schooling. This present study provides clear evidence that children developed relational reasoning which was supported by early multiplicative foundations of fractions through a spatial reasoning approach to learning fractions. Therefore, there is an opportunity to extend this research to examine how this innovative approach supports children's transition from a spatial reasoning approach to formal symbolic understanding later in primary school and beyond.

8.5.3 Exploring the Potential of Gesture

The children's spontaneous use of gesture is an important finding of this study. As described in the previous chapters, the way children spontaneously gestured when describing their reasoning or manipulating materials provided a critical lens in which to interpret how they were establishing the fraction ideas and utilising spatial reasoning. The mechanism for how gesture supports young children's learning is also not well understood in relation to mathematics in general (Aldugom et al., 2020). As reported in Chapter Two, there is very little literature concerning the role of gesture and fraction understanding. The studies that do exist, predominantly focus on how children gesture in response to fraction tasks in experimental conditions, or how teacher led gestures assist in learning mathematics. A video study would be an appropriate methodology to explore the ways in which a spatial reasoning approach to learning fractions evokes the use of gesture.

8.6 Final Reflection

This research serves as a reminder of how mathematically capable young children are, and that we must not underestimate their creative and intuitive ways of thinking and communicating. It highlights that for learning fractions, with the right instruction, children can understand complex ideas well before previously suggested, and that instruction should be derived from a multiplicative, partitioning approach. The findings have revealed that opportunities to engage with explicit spatial reasoning strategies when exploring early fraction as operator, fraction as a measure and fraction as a relation ideas are not only beneficial but are critical to assist children to make connections between these mathematical ideas. As educators, we must strive to provide learning environments that consistently enhance the cognitive skills and intuition of children.

This study extends the current perspectives related to how children develop early rational number concepts. In doing so, it has provided a platform for demonstrating the affordances of other cognitive domains – such as specific spatial skills and abilities, and how they enhance and can be embedded into children's thinking and learning of rational number. This innovative approach provides teachers with clear and practical ways to support their young students, which I hope, builds their professional capabilities and self-efficacy for teaching this area of mathematics. Finally, but most importantly, it is my hope that this research will contribute to an improvement in children's understanding of fractions by enabling them to fulfill their learning potential in this area. When we embrace the powerful and creative ways in which young children think and communicate mathematically, our profession will ultimately provide greater opportunities for young children's mathematical success at school and beyond.

References

Abrahamson, D. (2012). Rethinking intensive quantities via guided mediated abduction. *Journal of the Learning Sciences*, *21*(4), 626–649. https://doi.org/10.1080/10508406.2011.633838

Aldugom, M., Fenn, K. & Cook, S. W. (2020). Gesture during math instruction specifically benefits learners with high visuospatial working memory capacity. *Cognitive Research*, *5*(27). https://doi.org/10.1186/s41235-020-00215-8

Alibali, M. W. (2005). Gesture in spatial cognition: Expressing, communicating, and thinking about spatial information. *Spatial Cognition and Computation*, *5*(4), 307–331. https://doi.org/10.1207/s15427633scc0504_2

Alibali, M. W., Kita, S., & Young, A. J. (2000). Gesture and the process of speech production: We think, therefore we gesture. *Language and Cognitive Processes*, *15*(6), 593– 613. https://doi.org/10.1080/016909600750040571

Alibali, M. W., Nathan, M. J., Wolfgram, M. S., Church, R. B., Jacobs, S. A., Johnson Martinez, C., & Knuth, E. J. (2014). How teachers link ideas in mathematics instruction using speech and gesture: A corpus analysis. *Cognition and Instruction, 32*(1), 65– 100. https://doi.org/10.1016/j.dr.2014.10.001

Amato, S. A. (2005). Developing students' understanding of the concept of fractions as numbers. In H. L. Chick, & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 49– 56). University of Melbourne. http://onlinedb.terc.edu

Anderson, T. & Shattuck, J. (2012). Design-based research: A decade of progress in education research? *Educational Researcher, 41*(1), 16–25. https://doi.org/10.3102/0013189X11428813

Anderson-Pence, K. L., Moyer-Packenham, P. S., Westenskow, A., Shumway, J., & Jordan, K. (2014). Relationships between visual static models and students' written solutions to fraction tasks. *International Journal for Mathematics Teaching and Learning, 15,* 1– 18. https://www.academia.edu/download/45293621/Relationships_between_visual_static_mode20160502-31624-8dzm5q.pdf

Anney, V.N. (2014) Ensuring the Quality of the Findings of Qualitative Research: Looking at Trustworthiness Criteria. *Journal of Emerging Trends in Educational Research and Policy Studies, 5*, 272-281. https://www.researchgate.net/profile/Vicent-Anney/post/How-are-validity-and-reliability-issues-dealt-with-in-qualitative-studies/attachment/59d61daa79197b8077979180/AS%3A272153998495749%401441897977647/download/Ensuring+the+quality+of+the+findings+of+qualitative+research+%284%29.pdf

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*(3), 215–241. https://doi.org/10.1023/A:1024312321077

Australian Curriculum Assessment and Reporting Authority [ACARA]. (n.d). *Foundation to Year 10 curriculum: Mathematics, 8.4.* https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

Australian Curriculum, Assessment and Reporting Authority [ACARA]. (2023). *F– 10 Curriculum: Mathematics, 9.0*. https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/foundation-year_year-1_year-2_year-3_year-4_year-5_year-6_year-7_year-8_year-9_year-10?view=quick&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&side-by-side=1&strands-start-index=0&subjects-start-index=0

Bailey, D. H., Hoard, M. K., Nugent, L. & Geary, D. C. (2012). Competence with fractions predicts gains in mathematics achievement. *Journal of Experimental Child Psychology*, *113*(3), 447–455. http://dx.doi.org/10.1016/j.jecp.2012.06.004

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, *93*, 373–397. http://links.jstoLorg/sici?sici=0013-5984%28199303%2993%3A4%3C373%3A W AEOTM%3E2.0.CO%3B2-F

Barab, S. & Squire, K. (2004). Design-based research: Putting a stake in the ground. *The Journal of the Learning Sciences*, *13*(1), 1–14. https://doi.org/10.1207/s15327809jls1301_1

Barth, H., Baron, A., Spelke, E., & Carey, S. (2009). Children's multiplicative transformations of discrete and continuous quantities. *Journal of Experimental Child Psychology*, *103*(4), 441–454. https://doi.org/10.1016/j.jecp.2009.01.014

Baroody, A. J. (1989). One point of view: Manipulatives don’t come with guarantees. *The Arithmetic Teacher*, *37*(2), 4–5. https://doi.org/10.5951/AT.37.2.0004

Baroody, A. J., & Benson, A. (2001). Early Childhood Corner: November 2001: Early Number Instruction. *Teaching Children Mathematics, 8*(3), 154– 158. link.gale.com/apps/doc/A80352787/AONE?u=unisa&sid=googleScholar&xid=e9e3a15d.

Baroody, A. J. & Hume, J. (1991). Meaningful mathematics instruction: The case of fractions. *Remedial and Special Education*, *12*(3), 54–68. https://doi.org/10.1177/074193259101200307

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, (pp. 843 – 908). NCTM/ Information Age Publishing.

Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education, 27*(3), 258–292. https://doi.org/10.2307/749365

Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. V. A. (1998). Students' spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education*, *29*(5), 503– 532. https://www.jstor.org/stable/749731

Baturo, A. R. & Cooper, T. J. (1999). Fractions, reunitisation and the number-line representation. In O. Zaslavsky (Ed.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2 pp. 95-103). PME. https://files.eric.ed.gov/fulltext/ED489632.pdf

Behr, M. J., Harel, G., Post, T. & Lesh, R. (1992). Rational number, ratio, proportion. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 296–333). Macmillan.

Behr, M. J., Harel, G., Post, T. & Lesh, R. (1993). Rational numbers: Toward a semantic analysis – emphasis on the operator construct. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational Numbers: An Integration of Research* (pp. 13–47). Lawrence Erlbaum.

Behr, M. J., Lesh, R., Post, T. R. & Silver, E. A. (1983). Rational-number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 91–126). Academic Press.

Behr, M. J, Lesh, R. & Post, T. (1981). Rational number ideas and the role of representational systems [Paper presentation]. Annual Meeting of the American Educational Research Association, Los Angeles, CA. https://files.eric.ed.gov/fulltext/ED202692.pdf

Behr, M. J., Wachsmuth, I., Post, T. R. & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education*, *15*(5), 323–341. https://doi.org/10.2307/748423

Begolli, K. N., Booth, J. L., Holmes, C. A., & Newcombe, N. S. (2020). How many apples make a quarter? The challenge of discrete proportional formats. *Journal of Experimental Child Psychology*, *192*, 104774. https://doi.org/10.1016/j.jecp.2019.104774

Beilstein, S. O. (2019). *Supporting children's conceptual understanding of fractions with manipulatives and gesture* [Doctoral dissertation]. University of Illinois at Urbana-Champaign. http://hdl.handle.net/2142/104770

Berch, D. (2016). Disciplinary differences between cognitive psychology and mathematics education: A developmental disconnection syndrome. Reflections on 'Challenges in Mathematical Cognition' by Alcock et al., (2016). *Journal of Numerical Cognition*, *2*(1), 42– 47. https://doi.org/10.5964/jnc.v2i1.23

Biddlecomb, B. D. (2002). Numerical knowledge as enabling and constraining fraction knowledge: An example of the reorganization hypothesis. *The Journal of Mathematical Behaviour*, *21*(2), 167–190. https://doi.org/10.1016/S0732-3123(02)00117-7

Bitsch, V. (2005). Qualitative research: A grounded theory example and evaluation criteria. *Journal of Agribusiness*, *23*(1), 75– 91. https://ageconsearch.umn.edu/record/59612/

Bloom, P., & Wynn, K. (1997). Linguistic cues in the acquisition of number words. *Journal of Child Language*, *24*(3), 511– 533. https://doi.org/10.1017/S0305000997003188

Blumer, H. (1954). What is wrong with social theory?. *American Sociological Review, 19*(1), 3–10. https://doi.org/10.2307/2088165

Bobis, J. (2008). Early spatial thinking and the development of number sense. *Australian Primary Mathematics Classroom, 13*(3), 4– 9. https://search.informit.org/doi/pdf/10.3316/informit.412415906852030

Bobis, J., Way, J. (2018). Building connections between children's representations and their conceptual development in mathematics. In V. Kinnear, M. Y. Lai, T. Muir (Eds.), *Forging connections in early mathematics teaching and learning*, (pp. 55– 72). Springer. <ahref="http://dx.doi.org/10.1007/978-981-10-7153-9_4">[More Information]

Bobos-Kristof, G. (2015). *Teaching fractions through a measurement approach to prospective elementary teachers: A design experiment in a math methods course* [Doctoral dissertation], Concordia University.

Booth, J. L. & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper’s doorman? *Contemporary Educational Psychology, 37*(4), 247–253. https://doi.org/10.1016/j.cedpsych.2012.07.001

Booth, J. L., & Siegler, R. S. (2006). Developmental and individual differences in pure numerical estimation. *Developmental Psychology*, *42*(1), 189–201. https://doi.org/10.1037/0012-1649.41.6.189

Booth, J. L. & Siegler, R. S. (2008). Numerical magnitude representations influence arithmetic learning. *Child Development*, *79*(4), 1016–1031. https://doi.org/10.1111/j.1467-8624.2008.01173.x

Boyatzis, R. E. (1998). *Transforming qualitative information: Thematic analysis and code development*. Sage Publications.

Boyer, T. W. & Levine, S. C. (2012). Child proportional scaling: Is 1/3= 2/6= 3/9= 4/12? *Journal of Experimental Child Psychology*, *111*(3), 516–533. https://doi.org/10.1016/j.jecp.2011.11.001

Boyer, T. W., Levine, S. C. & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong. *Developmental Psychology*, *44*(5), 1478–1490. https://doi.org/10.1037/a0013110

Braithwaite, D. W. & Siegler, R. S. (2018). Developmental changes in the whole number bias. *Developmental Science*, *21*(2). https://doi.org/10.1111/desc.12541

Brannon, E. M., Lutz, D. & Cordes, S. (2006). The development of area discrimination and its implications for number representation in infancy. *Developmental Science*, *9*(6), 59–64. https://doi.org/10.1111/j.1467-7687.2006.00530.x

Brannon, E. M., Suanda, S. & Libertus, K. (2007). Temporal discrimination increases in precision over development and parallels the development of numerosity discrimination. *Developmental Science*, *10*(6), 770–777. https://doi.org/10.1111/j.1467-7687.2007.00635.x

Braun, V. & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Brizuela, B. M. (2006). Young children’s notations for fractions. *Educational Studies in Mathematics*, *62*, 281–305. http://doi.org/10.1007/s10649-005-9003-3

Brousseau, G., Brousseau, N. & Warfield, V. (2004). Rationals and decimals as required in the school curriculum: Part 1: Rationals as measurement. *Journal of Mathematical Behaviour*, *23*(1), 1–20. https://doi.org/10.1016/j.jmathb.2003.12.001

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *Journal of the Learning Sciences*, *2*(2), 141–178. https://doi.org/10.1207/s15327809jls0202_2

Brown, J. P. (2019). Real-world task context: meanings and roles. In G. A. Stillman, & J. Brown (Eds.), *Lines of inquiry in mathematical modelling research in education*, 53– 81. Springer. https://doi.org/10.1007/978-3-030-14931-4_4

Bruce C. D. (2013). *Math Teaching for Learning: Developing Proficiency with Partitioning, Iterating and Disembedding.* http://www.edugains.ca/resourcesMath/CE/LessonsSupports/Fractions/SupportDocs/DevelopingProficiencyWithPartitioningIteratingAndDisembedding_AODA.pdf

Bruce, C., Bennett, S. & Flynn, T. (2014). *Fractions Operations: Multiplication and Division Literature Review.* http://www.edugains.ca/resourcesMath/CE/LessonsSupports/Fractions/FractionsOperations_MultDiv_AODA.pdf

Bruce, C., Bennett, S., Flynn, T., Yearley, S & Lakelands, T. (2015a). *Rational Number Teaching and Learning – Literature Review.* https://www.fractionslearningpathways.ca/pdf/Rational_Number_LR_15May2018_FINAL.pdf

Bruce, C., Chang, D., Flynn, T. & Yearley, S. (2013). *Foundations to learning and teaching fractions: Addition and subtraction*. http://www.edugains.ca/resourcesDP/Resources/PlanningSupports/FINALFoundationstoLearningandTeachingFractions.pdf

Bruce, C. D., Davis, B., Sinclair, N., McGarvey, L., Hallowell, D., Drefs, M., Francis, K., Hawes, Z., Moss, J., Mulligan, J., Okamoto, Y., Whiteley, W. & Woolcott, G. (2017). Understanding gaps in research networks: Using ‘spatial reasoning’ as a window into the importance of networked educational research. *Educational Studies in Mathematics, 95*(2), 143–161. http://doi.org/10.1007/s10649-016-9743-2

Bruce, C., Flynn, T., & Bennett, S. (2016). A focus on exploratory tasks in lesson study: The Canadian ‘Math for young children’ project, *ZDM – Mathematics Education, 48*(4), 541–554.
https://doi.org/ 10.1007/s11858-015-0747-7

Bruce, C. D., & Hawes, Z. (2015a). The role of 2D and 3D mental rotation in mathematics for young children: what is it? Why does it matter? And what can we do about it?, *ZDM–Mathematics Education, 47*, 331-343. https://doi.org/10.1007/s11858-014-0637-4

Bruce, C. D., Sinclair, N., Moss, J., Hawes, Z., & Caswell, B. (2015b). Spatialising the curriculum. In B. Davis (Ed.), *Spatial reasoning in the early years* (pp. 85– 106). Routledge. https://doi.org/10.4324/9781315762371

Bruner, J. S. (1966). *Toward a theory of instruction*. Harvard University Press.

Bruner, J. S. (1973). *Beyond the information given: Studies in the psychology of knowing.* W. W. Norton.

Callingham, R. & Siemon, D. (2021). Connecting multiplicative thinking and mathematical reasoning in the middle years. *The Journal of Mathematical Behavior*, *61*, https://doi.org/10.1016/j.jmathb.2020.100837

Cannon, J., Levine, S. & Huttenlocher, J. (2007). A system for analyzing children and caregivers' language about space in structured and unstructured contexts. *Spatial Intelligence and Learning Center, (SILC*). https://www.silc.northwestern.edu/spatial-language-coding-manual/

Carraher, D. W. & Schliemann, A. D. (1991). Children's understanding of fractions as expressions of relative magnitude. In F. Furinghetti, (Ed.), *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 184–191). PME. https://www.igpme.org/wp-content/uploads/2019/05/PME15-1991-Assisi.zip

Carraher, D. W. (1993). Lines of thought: A ratio and operator model of rational number. *Educational Studies in Mathematics*, *25*(4), 281– 305. https://link.springer.com/content/pdf/10.1007/BF01273903.pdf

Carraher, D. W. (1996). Learning about fractions. In L. P. Steffe, P. Nesher, P. Cobb, G. A Goldin & B. Greer (Eds.), *Theories of Mathematical Learning* (pp. 241–266). Taylor and Francis.https://www.researchgate.net/profile/David_Carraher/publication/313744018_Learning_about_fractions/links/5ce2fcf9458515712eb87dd9/Learning-about-fractions.pdf

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, *37*(2), 87–115. https://doi.org/ 10.2307/30034843

Carroll, J. B. (1993). *Human cognitive abilities: A survey of factor-analytic studies*. Cambridge University Press. https://doi.org/10.1017/CBO9780511571312

Casasanto, D., & Bottini, R. (2014). Spatial language and abstract concepts. Wiley Interdisciplinary Reviews, *Cognitive Science, 5*(2), 139– 149. https://doi.org/10.1002/wcs.1271

Casasola, M., Wei, W. S., Suh, D. D., Donskoy, P. & Ransom, A. (2020). Children’s exposure to spatial language promotes their spatial thinking. *Journal of Experimental Psychology: General*, *149*(6), 1116–1136. https://doi.org/10.1037/xge0000699

Cartwright, K. (2019). “Because 7 and 8 are always in all of them”: What do students write and say to demonstrate their mathematical fluency? In G. Hine, S. Blackley, & A. Cooke (Eds.), *Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia*, (pp. 156– 163). MERGA https://merga.net.au/Public/Publications/Annual_Conference_Proceedings/2019-MERGA-conference-proceedings.aspx

Charalambous, C. Y. & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students’ understandings of fractions. *Educational Studies in Mathematics*, *64*(3), 293–316. https://doi.org/10.1007/s10649-006-9036-2

Chatterjee, A. (2008). The neural organization of spatial thought and language. *Seminars in Speech and Language*, *29*(3), 226–238. https://doi.org/10.1055/s-0028-1082886

Cheng, Y. L. & Mix, K. S. (2014). Spatial training improves children’s mathematics ability. *Journal of Cognition and Development*, *15*(1), 2–11. https://files.eric.ed.gov/fulltext/ED580905.pdf

Chow, J. C., & Jacobs, M. (2016). The role of language in fraction performance: A synthesis of literature. *Learning and Individual Differences, 47*, 252-257. https://doi.org/10.1016/j.lindif.2015.12.017

Clarke, D. (2006). Fractions as division: The forgotten notion? In J. Way & J. M. Bobis (Eds.), *Fractions: Teaching for understanding* (pp. 33–41). Australian Association of Mathematics Teachers. https://primarystandards.aamt.edu.au/Journals

Clarke, V. & Braun, V. (2013). Teaching thematic analysis: Overcoming challenges and developing strategies for effective learning. *The Psychologist*, *26*(2). http://www.thepsychologist.org.uk/archive/archive_home.cfm?volumeID=26&editionID=222&ArticleID=2222

Clarke, D. M. & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics*, *72*(1), 127–138. https://doi.org/10.1007/s10649-009-9198-9

Clarke, D. M., Roche, A. & Mitchell, A. (2007). Year six fraction understanding: A part of the whole story. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 207–216). MERGA. https://merga.net.au/Public/Publications/Annual_Conference_Proceedings/2007_MERGA_CP.aspx

Clarke, D. M., Roche, A., & Mitchell, A. (2008). 10 practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School, 13*(7), 372– 380. https://opencuny.org/citycollegespring2015/files/2015/02/Practical-tips-for-Fracations.pdf

Clarke, D., Roche, A. & Mitchell, A. (2011). One-to-one student interviews provide powerful insights and clear focus for the teaching of fractions in the middle years. In J. Way & J. M. Bobis (Eds.), *Fractions: Teaching for understanding* (pp. 23–41). Australian Association of Mathematics Teachers. https://primarystandards.aamt.edu.au/content/download/19922/272983/file/tdt_F_clarke1.pdf

Clarke, D. M., Roche, A., Mitchell, A. & Sukenik, M. (2006). Assessing student understanding of fractions using task-based interviews. In J. Novotna, H. Maraova & M. Kratka. (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (pp. 337–344). PME. https://www.igpme.org/wp-content/uploads/2019/05/PME30-2006-Prag.zip

Clements, D. H. (1999). Subitizing: What is it? Why teach it? *Teaching Children Mathematics, 5*, 400–405. https://pdfs.semanticscholar.org/1922/5ee2abccc971235863742962dd971382a2b1.pdf?_ga=2.262728204.1861820788.1580347137-22989661.1543977718

Clements, D. H. & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 420–464). Macmillan Publishing.

Clements, M. A. & Del Campo, G. (1987). Fractional understanding of fractions: Variations in children's understanding of fractional concepts across embodiments, Grades 2 through 5. In J. D. Novak (ed.), *Proceedings of the Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics* (vol. 3, pp. 98–110). Cornell University.

Clements, D. H. & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach*. Routledge.

Clements, D. H., & Sarama, J. (2017/2019). *Learning and Teaching with Learning Trajectories [LT]2.* Marsico Institute, Morgridge College of Education, University of Denver. https://www.learningtrajectories.org/

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23*(7), 13–20. https://doi.org/10.3102/0013189X023007013

Cobb, P. (1995). Cultural Tools and Mathematical Learning: A Case Study. *Journal for Research in Mathematics Education, 26*(4), 362–385. https://doi.org/10.2307/749480

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R. & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13. https://doi.org/10.3102/0013189X032001009

Cobb, P. & Gravemeijer, K. (2014). Experimenting to support and understand learning processes. In A. E. Kelly, R. A. Lesh & J. Y. Baek (Eds.), *Handbook of design research methods in education* (pp. 86–113). Routledge.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences, 10*(1– 2), 113– 163. https://doi.org/10.1207/S15327809JLS10-1-2_6

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3– 4), 175– 190. https://files.eric.ed.gov/fulltext/ED389535.pdf

Cohen, J. (2013). *Statistical power analysis for the behavioural sciences*. Academic Press. http://dx.doi.org/10.1007/978-3-642-77750-9_2

Cohen, L., Manion, L. & Morrison, K. (2007). *Research methods in education* (6th ed.). Routledge Falmer.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 292–332). New York Press.

Confrey, J. (2006). The evolution of design studies as methodology. In K. R. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 135– 152). New York: Cambridge University Press.

Confrey, J. (2008). *Learning Trajectories and Rational Number Reasoning* [PowerPoint slides]. https://www.human.cornell.edu/sites/default/files/HD/nsfalw/Confrey-NSF.pdf

Confrey, J. (2012). Better measurement of higher cognitive processes through learning trajectories and diagnostic assessments in mathematics: The challenge in adolescence. In V. F. Reyna, S. B. Chapman, M. R. Dougherty & J. Confrey (Eds.), *The adolescent brain: Learning, reasoning, and decision making* (pp. 155–182). American Psychological Association. https://doi.org/10.1037/13493-006

Confrey J. & Lachance J. (2000). Transformative teaching experiments through conjecture-driven research design. In Kelly A.E. & Lesh R.A. (Eds.), *Handbook of research design in mathematics and science education,* (pp. 231–265). Lawrence Erlbaum Associates. https://eric.ed.gov/?id=ED478316

Confrey, J. & Maloney, A. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory. In K. Gomez, L. Lyons & J. Radinsky (Eds.), *Learning in the disciplines: Proceedings of the 9th International Conference of the Learning Sciences* (pp. 968–975). International Society of the Learning Sciences. https://doi.dx.org/10.22318/icls2010.1.968

Confrey, J., & Maloney, A. (2015). A design research study of a curriculum and diagnostic assessment system for a learning trajectory on equipartitioning. *ZDM – Mathematics Education, 47*, 919-932. https://doi.org/10.1007/s11858-015-0699-y

Confrey, J., Maloney, A. P. & Corley, A. K. (2014a). Learning trajectories: A framework for connecting standards with curriculum. *ZDM – Mathematics Education*, *46*(5), 719–733. https://doi.dx.org/10.1007/s11858-014-0598-7

Confrey, J., Maloney, A., Nguyen, K., Mojica, G. & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Thessaloniki, Greece.

Confrey, J., Maloney, A. P., Nguyen, K. H. & Rupp, A. A. (2014b). Equipartitioning, a foundation for rational number reasoning. In A. Maloney, J. Confrey & K. H. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education* (pp. 61–96). Information Age Publishing.

Confrey, J., Maloney, A. P., Wilson, P. H., & Nguyen, K. H. (2010). Understanding over time: The cognitive underpinnings of learning trajectories. *In Annual Meeting of the American Education Research Association*, Denver, CO. https://www.aera.net/Portals/38/docs/Meetings_and_Events/AM2010/ProgramSchedule.pdf

Confrey, J., & Scarano, G. H. (1995). Splitting Reexamined: Results from a Three-Year Longitudinal Study of Children in Grades Three to Five. In D. T. Owens (Ed.), *Proceedings of the 17th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 421– 426). PME. https://files.eric.ed.gov/fulltext/ED389583.pdf

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, *26*(1), 66–86. https://doi.org/10.2307/749228

Congdon, E. L., Novack, M. A., Brooks, N., Hemani-Lopez, N., O'Keefe, L., & Goldin-Meadow, S. (2017). Better together: Simultaneous presentation of speech and gesture in math instruction supports generalization and retention. *Learning and Instruction, 50,* 65– 74. https://doi.org/10.1016/j.learninstruc.2017.03.005

Cordes, S. & Brannon, E. M. (2009). Crossing the divide: Infants discriminate small from large numerosities. *Developmental Psychology*, *45*(6), 1583. https://doi.dx.org/10.1037/a0015666

Corley, A. K. (2013). *A design study of co-splitting as situated in the equipartitioning learning trajectory*. [Doctoral dissertation]. North Carolina State University.

Cortina, J. L., Visnovska, J. & Zúñiga, C. (2014). Equipartition as a didactical obstacle in fraction instruction. *Acta Didactica Universitatis Comenianae Mathematics*, *14*(1), 1–18. https://www.researchgate.net/publication/282264894_Equipartition_as_a_didactical_obstacle_in_fraction_instruction

Cramer, K. A., Behr, M. J., Post, T. R. & Lesh, R. (1997). *Rational number project: Fraction lessons for the middle grades – Level 1*. Kendall Hunt Publishing.

Cramer, K. A., Post, T. R. & delMas, R. C. (2002). Initial fraction learning by fourth-and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the rational number project curriculum. *Journal for Research in Mathematics Education*, *33*(2), 111–144. https://doi.org/10.2307/749646

Cramer, K. & Wyberg, T. (2009). Efficacy of different concrete models for teaching the part-whole construct for fractions. *Mathematical Thinking and Learning*, *11*(4), 226–257. https://doi.org/10.1080/10986060903246479

Creswell, J. W. (2003). *Research design: Qualitative, quantitative, and mixed methods* (2nd ed.). Sage Publications.

Creswell, J. W. (2013). *Qualitative inquiry & research design: Choosing among five approaches* (3rd ed.). Sage Publications.

Daly, J., Kellehear, A. & Gliksman, M. (1997). *The public health researcher: A methodological approach.* Oxford University Press.

Davydov, V. V., & Tsvetkovich, Z. H. (1991). On the objective origin of the concept of fractions. *Focus on Learning Problems in Mathematics, 13*(1), 13– 64. https://doi.org/10.1027/28893568480701560

De Hevia, M. D. & Spelke, E. S. (2010). Number-space mapping in human infants. *Psychological Science*, *21*(5), 653–660. https://doi.org/10.1177/0956797610366091

De Ruiter, J. P. (1998). *Gesture and speech production*. [Ph.D. dissertation] University of Nijmegen. https://pure.mpg.de/rest/items/item_2057686_6/component/file_2273189/content

De Villiers, M. R. (2005). Interpretive research models for Informatics: Action research, grounded theory, and the family of design-and development research. *Alternation*, *12*(2), 10–52. https://www.ingentaconnect.com/content/sabinet/alt/2005/00000012/00000002/art00008

DeCuir-Gunby, J. T., Marshall, P. L. & McCulloch, A. W. (2011). Developing and using a codebook for the analysis of interview data: An example from a professional development research project. *Field Methods*, *23*(2), 136–155. https://doi.org/10.1177/1525822X10388468

DeLoache, J. S. (1987). Rapid change in the symbolic functioning of very young children. *Science*, *238*(4833), 1556–1557. https://doi.org/10.1126/science.2446392

Department of Education, Employment and Workplace Relations [DEEWR], (2009). *Belonging, being and becoming: The early years learning framework for Australia.* https://doc.education.gov.au/system/files/doc/other/belonging_being_and_becoming_the_early_years_learning_framework_for_Australia_.PDF

Denzin, N. K., Lincoln, Y. S., & Giardina, M. D. (2006). Disciplining qualitative research. *International Journal of Qualitative Studies in Education, 19*(6), 769– 782. https://doi.org/10.1080/09518390600975990

Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher*, *32*(1), 5–8. https://doi.org/10.3102/0013189X032001005

di Sessa, A. & Cobb, C. (2004). Ontological innovation and the role of theory in design experiments. *Journal of the Learning Sciences, 13*(1). 77– 103. http://doi.org/10.1207/s15327809jls1301_4

Diezmann, C., & Lowrie, T. (2008). The role of information graphics in mathematical proficiency. In M., Goos, R., Brown, R, & K., Makar, (Eds.) *Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 647– 650). Mathematics Education Research Group of Australasia. https://eprints.qut.edu.au/18535/1/c18535.pdf

Dole, S. L. (1999). *Percent knowledge: Effective teaching for learning, relearning, and unlearning* [Doctoral dissertation]. Queensland University of Technology. https://eprints.qut.edu.au/36602/7/36602_Digitised%20Thesis.pdf

Dole, S., Clarke, D., Wright, T. & Hilton, G. (2012). Students' proportional reasoning in mathematics and science. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 195–202). PME. https://www.igpme.org/wp-content/uploads/2019/05/PME36-2012-Taipei.zip

Dole, S., Wright T., & Clarke, D (n.d.). *Proportional reasoning: Making connections between science and mathematics.* [PowerPoint Slides]. https://studylib.net/doc/25421625/proportional-reasoning

Downs, R. M. (1985). The representation of space: Its development in children and in cartography. In R. Cohen (Ed.), *The development of spatial cognition* (pp. 323–345). Erlbaum. https://doi.org/10.4324/9780203781654

Duffy, S., Huttenlocher, J. & Levine, S. (2005). It is all relative: How young children encode extent. *Journal of Cognition and Development*, *6*(1), 51–63. https://doi.org/10.1207/s15327647jcd0601_4

Education Services Australia, (n.d.). *Scootle*. https://www.scootle.edu.au/ec/p/home

Edwards, L. D. (2008). Conceptual integration, gesture, and mathematics. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda, (Eds.), *Proceedings of the Joint Meeting of the 32nd Conference of the International Group for the Psychology of Mathematics Education, and the North American Chapter* (vol. 2, pp. 423– 430). PME. https://www.pmena.org/pmenaproceedings/PMENA%2030%202008%20Proceedings%20Vol%202.pdf#page=434

Edwards, L. D. (2009). Gestures and conceptual integration in mathematical talk. *Educational Studies in Mathematics, 70*, 127– 141. https://doi.org/10.1007/s10649-008-9124-6

Edwards, L. D., & Robutti, O. (2014). Embodiment, modalities, and mathematical affordances. In L. Edwards, F. Ferrara, & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment in mathematics*, 1-23. Information Age Publishing.

Elia, I., & Evangelou, K. (2014). Gesture in a kindergarten mathematics classroom. European Early Childhood Education Research Journal, 22(1), 45–66. https://doi.org/10.1080/1350293X.2013.865357

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction*, *17*(3), 283–342. https://doi.org/10.1207/S1532690XCI1703_3

Empson, S. B. (2001). Equal sharing and the roots of fraction equivalence. *Teaching Children Mathematics*, *7*(7), 421–421. https://go.gale.com/ps/i.do?id=GALE%7CA72294541&sid=googleScholar&v=2.1&it=r&linkaccess=fulltext&issn=10735836&p=AONE&sw=w&userGroupName=unisa

Empson, S. B., Junk, D., Dominguez, H., & Turner, E. (2006). Fractions as the coordination of multiplicatively related quantities: A cross-sectional study of children's thinking. *Educational Studies in Mathematics*, *63*, 1-28. https://doi.org/ 10.1007/s10649-005-9000-6

Etikan, I., Musa, S. A., & Alkassim, R. S. (2016). Comparison of convenience sampling and purposive sampling. *American Journal of Theoretical and Applied Statistics, 5*(1), 1– 4. https://.doi.org/10.11648/j.ajtas.20160501.11

Fereday, J. & Muir-Cochrane, E. (2006). Demonstrating rigor using thematic analysis: A hybrid approach of inductive and deductive coding and theme development. *International Journal of Qualitative Methods*, *5*(1), 80–92. https://doi.org/10.1177/160940690600500107

Fischer, M. H. (2003). Cognitive representation of negative numbers. *Psychological Sciences*, *14*(3), 278–282. https://doi.org/10.1111/1467-9280.03435

Frick A. (2019). Spatial transformation abilities and their relation to later mathematics performance. *Psychological Research*, *83*(7), 1465–1484. https://doi.org/10.1007/s00426-018-1008-5

Frick, A., Hansen, M. A., & Newcombe, N. S. (2013). Development of mental rotation in 3– to 5– year-old children. *Cognitive Development, 28*(4), 386– 399. https://doi.org/10.1016/j.cogdev.2013.06.002

Frick, A., & Möhring, W. (2016). A matter of balance: Motor control is related to children's spatial and proportional reasoning skills. *Frontiers in Psychology, 6*, 2049. https://doi.org/10.3389/fpsyg.2015.02049

Frick, A. & Newcombe, N. S. (2012). Getting the big picture: Development of spatial scaling abilities. *Cognitive Development*, *27*(3), 270–282. https://doi.org/10.1016/j.cogdev.2012.05.004

Fowler, S., Cutting, C., Fiedler, S. H., & Leonard, S. N. (2022). Design-based research in mathematics education: Trends, challenges, and potential. *Mathematics Education Research Journal*, 1– 24. https://doi.org/10.1007/s13394-021-00407-5

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., Jordan, N. C., Siegler, R., Gersten, R. & Changas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology*, *105*(3), 683–700. https://doi.org/10.1037/a0032446

Fraenkel, J. R., Wallen, N. E., & Hyun, H. H. (2012). *How to design and evaluate research in educatio*n (8th ed.). Mc Graw Hill Education.

Freudenthal, H. (1973). *Mathematics as an educational task* (1st ed.). D. Reidel Publishing Company. https://doi.org/10.1007/978-94-010-2903-2

Gabriel, F. C., Coché, F., Szucs, D., Carette, V., Rey, B. & Content, A. (2013). A componential view of children's difficulties in learning fractions. *Frontiers in Psychology, 4*(715), 1–12. https://doi.org/10.3389/fpsyg.2013.00715

Geary, D. C. (2006). Development of mathematical understanding. In D. Kuhn, R. S. Siegler, W. Damon & R. M. Lerner (Eds.), *Handbook of child psychology: Cognition, perception, and language* (pp. 777–810). John Wiley & Sons.

Gentner, D. & Bowerman, M. (2009). Why some spatial semantic categories are harder to learn than others: The typological prevalence hypothesis. In J. Guo, E. Lieven, N. Budwig, S. Ervin-Tripp, K. Nakamura & Ş. Õzçalişkan (Eds.), *Crosslinguistic approaches to the psychology of language: Research in the tradition of Dan Isaac Slobin* (pp. 465–480). Psychology Press.

Gerofsky, S. (2014). Making sense of the multiple meanings of embodied mathematics learning. In Oesterle, S., Liljedahl, P., Nicol, C., & Allan, D. (Eds.) *Proceedings of the Joint Meeting of the International Group for the Psychology of Mathematics Education 38 and North American Chapter of the International Group for the Psychology of Mathematics Education*, (Vol. 3 pp. 145–152). https://files.eric.ed.gov/fulltext/ED599806.pdf

Getenet, S., & Callingham, R. (2017). Teaching fractions for understanding: Addressing interrelated concepts. In A. Downton, S. Livey & J. Hall (Eds.), *Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 227 – 284). MERGA. https://research.usq.edu.au/download/673ed6ebd04be5cce411556442dc981e52e3c101c3f7ed666cfa54dfd285b9be6/1218219/MERGA40%20_Getenet.pdf

Gilligan, K. A., Hodgkiss, A., Thomas, M. S. & Farran, E. K. (2018). The use of discrimination scaling tasks: A novel perspective on the development of spatial scaling in children. *Cognitive Development, 47,* 133–145. https://discovery.ucl.ac.uk/id/eprint/10050940/1/1-s2.0-S088520141730165X-main.pdf

Gilligan-Lee, K. A., Hawes, Z. C., & Mix, K. S. (2022). Spatial thinking as the missing piece in mathematics curricula. *Science of Learning, 7*(1), 10. https://doi.org/10.1038/s41539-022-00128-9

Goldin, G. A. (1997). Chapter 4: Observing mathematical problem solving through task-based interviews. *Journal for Research in Mathematics Education: Monograph*, *9,* 40–177. https://doi.org/10.2307/749946

Goldin, G. A. (1998). Representational systems, learning, and problem solving in mathematics. *The Journal of Mathematical Behavior*, *17*(2), 137–165. https://doi.org/10.1016/S0364-0213(99)80056-1

Goldin, G. A., & Janvier, C. (1998). Representations and the psychology of mathematics education. *The Journal of Mathematical Behavior, 17*(1), 1-4. https://www.researchgate.net/publication/269407907_A_joint_perspective_on_the_idea_of_representation_in_learning_and_doing_mathematics

Goldin, G. & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. A. Cuoco (Ed.), *The Roles of Representation in School Mathematics* (pp. 1–23). National Council of Teachers of Mathematics.

Goldkuhl, G. (2012). From action research to practice research. *Australasian Journal of Information Systems 17*(2). https://journal.acs.org.au/index.php/ajis/article/download/688/536

Goss, P., Hunter, J., Romanes, D., & Parsonage, H. (2015). *Targeted Teaching: How Better Use of Data Can Improve Student Learning.* Grattan Institute. https://grattan.edu.au/wp-content/uploads/2015/07/827-Targeted-Teaching.pdf

Goswami, U. (1989). Relational complexity and the development of analogical reasoning. *Cognitive Development*, *4*(3), 251–268. https://doi.org/10.1016/0885-2014(89)90008-7

Gould, P. (2005). Drawing sense out of fractions. In J. Way & J. M. Bobis (Eds.), *Fractions: Teaching for understanding* (pp. 133–138). Australian Association of Mathematics Teachers.

Gould, P. (2011). Developing an understanding of the size of fractions. In J. Way & J. M. Bobis (Eds.), *Fractions: Teaching for understanding* (pp. 63–70). Australian Association of Mathematics Teachers. https://www.aamt.edu.au/content/download/19933/273067/file/tdt_F_clarke2.pdf

Gould, P. (2013). Australia's next top fraction model. *Australian Primary Mathematics Classroom*, *18*(3), 5– 12. https://search.informit.org/doi/10.3316/informit.642802798480259

Gould, P., Outhred, L., & Mitchelmore, M. (2006). One-third is three-quarters of one-half. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Proceedings of the 29th Mathematics Education Research Group of Australia,* (pp. 262–269). MERGA.

Gravemeijer, K. (1998). Developmental research as a research method. In A. Sierpinska, & J. Kilpatrick, (Eds.), *Mathematics education as a research domain: A search for identity. New ICMI studies series,* (vol 4., pp. 277– 295). Springer. https://doi.org/10.1007/978-94-011-5470-3_18

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning*, *6*(2), 105– 128. https://doi.org/10.1207/s15327833mtl0602_3

Gravemeijer, K. & Van Eerde, D. (2009). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *The Elementary School Journal*, *109*(5), 510–524. https://doi.org/10.1086/596999

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, *22*(3), 170–218. https://doi.org/10.2307/749074

Guba, E. G. (1981). Annual review paper: Criteria for assessing the trustworthiness of naturalistic inquiries. *Educational Communication and Technology*, *29*(2), 75–91. http://www.jstor.org/stable/30219811

Gunderson, E. A., Ramirez, G., Beilock, S. L., & Levine, S. C. (2012). The relation between spatial skill and early number knowledge: the role of the linear number line. *Developmental Psychology*, *48*(5), 1229–1241. https://doi.org/10.1037/a0027433

Hackenberg, A. J. (2007). Units' coordination and the construction of improper fractions: A revision of the splitting hypothesis. *The Journal of Mathematical Behavior*, *26*(1), 27–47. https://cepa.info/fulltexts/764.pdf

Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior*, *32*(3), 538–563. https://doi.org/10.1016/j.jmathb.2013.06.007

Hackenberg, A. J., & Lee, M. Y. (2012). Pre-Fractional Middle School Students' Algebraic Reasoning. In L. R. Van Zoest, J. J. Lo & J. L. Kratky (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 943 – 950). PME. https://hdl.handle.net/2022/26819

Hackenberg, A. J., Norton, A. & Wright, R. J. (2016). *Developing fractions knowledge*. Sage Publications.

Hackenberg, A. J. & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior, 28*(1), 1–18. https://eric.ed.gov/?redir=http%3a%2f%2fdx.doi.org%2f10.1016%2fj.jmathb.2009.04.004

Halberda, J., Mazzocco, M. M. & Feigenson, L. (2008). Individual differences in non-verbal number acuity correlate with maths achievement. *Nature*, *455*(7213), 665–668. https://doi.org/10.1038/nature07246

Hallett, D., Nunes, T. & Bryant, P. (2010). Individual differences in conceptual and procedural knowledge when learning fractions. *Journal of Educational Psychology*, *102*(2), 395–406. https://doi.org/10.1037/a0017486

Hamre, B. K. & Pianta, R. C. (2007). Learning opportunities in preschool and early elementary classrooms. In R. C. Pianta, M. J. Cox & K. L. Snow (Eds.), *School readiness and the transition to kindergarten in the era of accountability* (pp. 49–83). Paul H. Brookes Publishing.

Hamre, B. K., Pianta, R. C. & Chomat-Mooney, L. (2009). Conducting classroom observations in school-based research. In L. M. Dinella (Ed.), *Conducting science-based psychology research in schools* (pp. 79–105). American Psychological Association. http://dx.doi.org/10.1037/11881-004

Hannula, M. S. (2003). Locating fractions on a number line. In N. A. Pateman, B. J. Dougherty, J. T Zilliox (Eds.), *Proceedings of the* 27th *International Group for the Psychology of Mathematics Education, 3,* 17– 24. https://files.eric.ed.gov/fulltext/ED500981.pdf

Hansen, N., Jordan, N. C., Fernandez, E., Siegler, R. S., Fuchs, L., Gersten, R. & Micklos, D. (2015). General and math-specific predictors of sixth graders' knowledge of fractions. *Cognitive Development*, *35*, 34–49. https://doi.org/10.1016/j.cogdev.2015.02.001

Hansen, N., Jordan, N. C. & Rodrigues, J. (2017). Identifying learning difficulties with fractions: A longitudinal study of student growth from third through sixth grade. *Contemporary Educational Psychology*, *50*, 45–59. https://doi.org/10.1016/j.cedpsych.2015.11.002

Harel, G., & Confrey, J. (Eds.). (1994). *Development of multiplicative reasoning in the learning of mathematics,* State University of New York Press.

Harris, D., Lowrie, T., Logan, T., & Hegarty, M. (2021). Spatial reasoning, mathematics, and gender: Do spatial constructs differ in their contribution to performance? *British Journal of Educational Psychology, 91*(1), 409–441. https://doi.org/10.1111/bjep.12371

Hart, C. (2017) *Knock, knock dinosaur*. Hodder Children's Books.

Hawes, Z., Moss, J., Caswell, B., & Poliszczuk, D. (2015). Effects of mental rotation training on children's spatial and mathematics performance: A randomized controlled study. *Trends in Neuroscience and Education, 4*(3), 60– 68. https://doi.org/10.1016/j.tine.2015.05.001

Hawes, Z., Sokolowski, H. M., Ononye, C. B., & Ansari, D. (2019). Neural underpinnings of numerical and spatial cognition: An fMRI meta-analysis of brain regions associated with symbolic number, arithmetic, and mental rotation. *Neuroscience & Biobehavioral Reviews, 103*, 316-336. https://doi.org/10.1016/j.neubiorev.2019.05.007

He, W., Yang, Y. & Gao, D. (2018). Proportional reasoning in 5-to 6-year-olds. *Journal of Cognition and Development*, *19*(4), 389–412. https://doi.org/10.1080/15248372.2018.1495218

Herrington, J. A., McKenney, S., Reeves, T. C. & Oliver, R. (2007). Design-based research and doctoral students: Guidelines for preparing a dissertation proposal. In C. Montgomerie & J. Seale (Eds.), *Proceedings of Ed Media 2007: World Conference on Educational Multimedia, Hypermedia & Telecommunications* (pp. 4089–4097). AACE. https://ro.ecu.edu.au/ecuworks/1612/

Hewson, P. W. & Thorley, N. R. (1989). The conditions of conceptual change in the classroom. *International Journal of Science Education*, *11*(5), 541–553. https://doi.org/10.1080/0950069890110506

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*(2), 371–406. https://doi.org/10.3102/00028312042002371

Hilton, A., Hilton, G., Dole, S. & Goos, M. (2016). Promoting middle school students' proportional reasoning skills through an ongoing professional development programme for teachers. *Educational Studies in Mathematics*, *92*(2), 193–219. https://doi.org/10.1007/s10649-016-9694-7

Hino, K., & Kato, H. (2019). Teaching whole-number multiplication to promote children's proportional reasoning: A practice-based perspective from Japan. *ZDM – Mathematics Education, 51*, 125–137. https://doi.org/10.1007/s11858-018-0993-6

Höffler, T. N. (2010). Spatial ability: Its influence on learning with visualizations—a meta-analytic review. *Educational Psychology Review, 22*(3), 245–269. https://www.jstor.org/stable/24830657?seq=1#metadata_info_tab_contents

Holloway, I. D. & Ansari, D. (2009). Mapping numerical magnitudes onto symbols: The numerical distance effect and individual differences in children's mathematics achievement. *Journal of Experimental Child Psychology, 103*(1), 17–29. https://doi.org/10.1016/j.jecp.2008.04.001

Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic Bulletin & Review, 15*, 495-514. https://doi.org/10.3758/PBR.15.3.495

Howell, D. C. (2010). *Statistical methods for psychology*, (7th ed.). Cengage Learning.

Huinker, D. (2002). Examining dimensions of fraction operation sense. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions: 2002 yearbook* (pp. 72–78). National Council of Teachers of Mathematics. https://doi.org/10.26711/007577152790065

Hunting, R. P. (1983). Alan: A case study of knowledge of units and performance with fractions. *Journal for Research in Mathematics Education*, *14*(3), 182–197. https://www.jstor.org/stable/748381?seq=1#metadata_info_tab_contents

Hunting, R. P., Davis, G., & Bigelow, J. C. (1991). Higher order thinking in young children's engagements with a fraction machine. *Early Fraction Learning*, 73–90. https://doi.org/10.1007/978-1-4612-3194-3_5

Hunting, R. P., Davis, G. & Pearn, C. A. (1996). Engaging whole-number knowledge for rational-number learning using a computer-based tool. *Journal for Research in Mathematics Education*, *27*(3), 354–379. https://www.jstor.org/stable/749369?seq=1#metadata_info_tab_contents

Hurrell, D. (2013). *Effectiveness Of Teacher Professional Learning: Enhancing the Teaching of Fractions in Primary Schools*, [Doctoral dissertation]. https://ro.ecu.edu.au/theses/596

Hutchins, P. (1989). *The doorbell rang.* Harper Collins Books.

Huttenlocher, J., Newcombe, N. S. & Vasilyeva, M. (1999). Spatial scaling in young children. *Psychological Science*, *10*(5), 393–398. https://journals.sagepub.com/doi/pdf/10.1111/1467-9280.00175

Janvier, C., Girardon, C. & Morand, J. (1993). Mathematical symbols and representations. In P. S. Wilson (Ed.) *Research ideas for the classroom: High school mathematics* (pp. 79–102). Macmillian Publishing Company.

Jirout, J. J., Holmes, C. A., Ramsook, K. A. & Newcombe, N. S. (2018). Scaling up spatial development: A closer look at children's scaling ability and its relation to number knowledge. *Mind, Brain, and Education*, *12*(3), 110–119. https://onlinelibrary.wiley.com/doi/pdf/10.1111/mbe.12182

Jirout, J. & Newcombe, N. S. (2018). How much as compared to what: Relative magnitude as a key idea in mathematics cognition. In S. Mix & T. Battista (Eds.), *Visualizing mathematics: The role of spatial reasoning in mathematical thought* (pp. 3–24). Springer. https://doi.org/10.1007/978-3-319-98767-5_1

Jones, C. M. (2017). *Mental representations of fractions and decimals: Differences, commonalities, and implications for understanding* [Doctoral dissertation]. University of Huddersfield. http://eprints.hud.ac.uk/id/eprint/34424/1/FINAL%20THESIS%20-%20Jones%2C%20Corinna.pdf

Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R. & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Child Psychology*, *116*(1), 45–58. https://files.eric.ed.gov/fulltext/ED552739.pdf

Jordan, N. C., Rodrigues, J., Hansen, N. & Resnick, I. (2017). Fraction development in children: Importance of building numerical magnitude understanding. In D. C. Geary, D. B. Berch, R. J. Ochsendorf & K. M. Koepke (Eds.), *Mathematical Cognition and Learning: Acquisition of Complex Arithmetic Skills and Higher–Order Mathematics Concepts.* Academic Press. https://doi.org/10.1016/B978-0-12-805086-6.00006-0

Kalra, P. B., Hubbard, E. M., & Matthews, P. G. (2020). Taking the relational structure of fractions seriously: Relational reasoning predicts fraction knowledge in elementary school children. *Contemporary Educational Psychology, 62*, 101896. https://doi.org/10.1016/j.cedpsych.2020.101896

Keijzer, R. & Terwel, J. (2002). Audrey's acquisition of fractions: A case study into the learning of formal mathematics. *Educational Studies in Mathematics*, *47*, 53–73. https://doi.org/10.1023/A:1017971912662

Kell, H. J., Lubinski, D. & Benbow, C. P. (2013). Who rises to the top? Early indicators. *Psychological Science*, *24*(5), 648–659. https://doi.org/10.1177/0956797612457784

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional. In R. A. Lesh & D. A. Bradbrand (Eds.), *Number and measurement. Papers from a research workshop* (pp. 101–151). Information Reference Centre, ERIC/IRC. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.994.9838&rep=rep1&type=pdf#page=108

Kieren, T. (1980). The rational number construct: Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125–149). ERIC/SMEAC. https://files.eric.ed.gov/fulltext/ED212463.pdf

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 162–181). National Council of Teachers of Mathematics. Lawrence Erlbaum.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Studies in mathematical thinking and learning. rational numbers: An integration of research* (pp. 49–84). Lawrence Erlbaum Associates.

Kieren, T. E. (1995). Creating spaces for learning fractions. In J. T. Sowder & B. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 31–65). State University of New York Press.

Kieren, T. (1999). Language Use in Embodied Action and Interaction in Knowing Fractions. In D. Owens, (Ed.), *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. https://files.eric.ed.gov/fulltext/ED466385.pdf

Kivunja, C., & Kuyini, A. B. (2017). Understanding and applying research paradigms in educational contexts. *International Journal of Higher Education, 6*(5), 26–41. https://files.eric.ed.gov/fulltext/EJ1154775.pdf

Kosslyn, S. M. (1988). Aspects of a cognitive neuroscience of mental imagery. *Science, 240*(4859), 1621–1626. https://doi.org/10.1126/science.3289115

Kullberg, A., & Björklund, C. (2020). Preschoolers' different ways of structuring part-part-whole relations with finger patterns when solving an arithmetic task. *ZDM – Mathematics Education*, *52,* 767–778. https://doi.org/10.1007/s11858-019-01119-8

Krause, C. M., & Salle, A. (2019). Towards cognitive functions of gestures: A case of mathematics. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.), *Proceedings of the 43rd Conference of the International Group for the Psychology in Mathematics Education* (pp. 496-503). PME. https://www.igpme.org/wp-content/uploads/2019/07/PME43-proceedings.zip

Krauss, R. M., Chen, Y., & Gotfexnum, R. F. (2000). 13 Lexical gestures and lexical access: a process model. In D. McNeill (Ed.), *Language and gesture* (pp. 261–283). Cambridge University Press.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education*, *27*(2), 170–193. https://www.jstor.org/stable/pdf/749599.pdf?refreqid=excelsior%3A7c6f7b26dc6854381 12c997db17c1190

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential knowledge and instructional strategies for teachers*. Lawrence Erlbaum.

Lamon, S. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco & F. Curcio (Eds.), *The roles of representations in school mathematics –2001 Yearbook* (pp. 146–165). National Council of Teachers of Mathematics.

Lamon, S. J. (2006). *Teaching fractions and ratios for understanding*. Lawrence Erlbaum.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–668). Information Age Publishing.

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (3rd ed.). Routledge. https://doi.org/10.4324/9780203803165

Lamon, S.J. (2020). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (4th ed.). Routledge. https://doi.org/10.4324/9781003008057

Landau, B. (2017). Update on “what” and “where” in spatial language: A new division of labour for spatial terms. *Cognitive Science, 41*, 321-350. https://doi.org/10.1111/cogs.12410

Landau, B., & Jackendoff, R. (1993). "What" and "where" in spatial language and spatial cognition. *Behavioural and Brain Sciences, 16*(2), 217–265. https://doi.org/10.1017/S0140525X00029733

Leiner, D. J. (2014). Convenience samples from online respondent pools: A case study of the SoSci Panel. *International Journal of Internet Science*, *20*(5), 1–18. https://www.researchgate.net/profile/Dominik-Leiner/publication/314190153_Leiner_2014_-_Convenience_Pools_from_Online_Respondent_Pools/data/58b968e392851c471d4a582a/Leiner-2014-Convenience-Pools-Working-Papernote.pdf

Leinhardt, G. (1988). Expertise in instructional lessons: An example from fractions. In D. A. Grouws & T. J. Cooney (Eds.), *Perspectives on research on effective mathematics teaching* (Vol. 1, pp. 47–66). NCTM.

Lesh, R., Post, T. R., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In R. Lesh, T. Post, M. Behr (Eds.), *Problems of representations in the teaching and learning of mathematics* (pp. 33–40). Lawrence Erlbaum.

Leslie, A. M., Gelman, R. & Gallistel, C. R. (2008). The generative basis of natural number concepts. *Trends in Cognitive Sciences*, *12*(6), 213–218. https://doi.org/10.1016/j.tics.2008.03.004

Lewis, M. R., Matthews, P. G. & Hubbard, E. M. (2016). Neurocognitive architectures and the nonsymbolic foundations of fractions understanding. In D. B. Berch, D. C. Geary & K. Mann Koepke (Eds.), *Development of mathematical cognition: neural substrates and genetic influences* (pp. 141–164). Elsevier Academic Press.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry.* Sage Publications.

Linn, M. C. & Petersen, A. C. (1985). Emergence and characterization of sex differences in spatial ability: A meta-analysis. *Child Development*, *56*(6), 1479–1498. https://doi.org/10.2307/1130467

Logan, T., & Greenlees, J. (2008). Standardised assessment in mathematics: The tale of two items. In M. Goos, Brown, R & K. Makar (Eds.), *Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australia* (pp. 655–658). Mathematics Education Research Group of Australia. https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=8b4a8663e7d7a2cbb4dc83e49c8d8f07bccda107#page=604

Logan, T., Lowrie, T., & Diezmann, C. M. (2014). Co-thought gestures: Supporting students to successfully navigate map tasks. *Educational Studies in Mathematics, 87*, 87–102. https://doi.org/10.1007/s10649-014-9546-2.

Lohman, D. F. (1988). Spatial abilities as traits, processes, and knowledge. In R. J. Sternberg (Ed.), *Advances in the psychology of human intelligence* (vol. 4, pp. 181–248). Lawrence Erlbaum Associates.

Lortie-Forgues, H., Tian, J. & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review*, *38*, 201–221. https://doi.org/10.1016/j.dr.2015.07.008

Lourenco, S. F. & Longo, M. R. (2009). Multiple spatial representations of number: Evidence for co-existing compressive and linear scales. *Experimental Brain Research*, *193*(1), 151–156. https://doi.org/10.1007/s00221-008-1698-9

Lowrie, T. (2010). Primary Students' decoding mathematics tasks: The role of spatial reasoning. *ACER Research Conference 2010 – Teaching Mathematics? Make it Count*. Retrieved from https://research.acer.edu.au/research_conference/RC2010/17august/4

Lowrie, T. (2012). Visual and spatial reasoning: The changing form of mathematics representation and communication. In B. Kaur & T. L. Toh (Eds.), *Reasoning, communication, and connections in mathematics: Yearbook 2012* (pp. 149–168). Association of Mathematics Educators. https://doi.org/10.1142/9789814405430_0008

Lowrie, T., & Diezmann, C. M. (2009). National numeracy tests: A graphic tells a thousand words. *Australian Journal of Education, 53*(2), 141–158. https://journals.sagepub.com/doi/pdf/10.1177/000494410905300204?casa_token=nof1Jdf5mjMAAAAA:E82l-kqMyNrhAQFX5UeRHjraVOj98uanx_0vnATvnImkG9yrICJQ9P9Je0VZmxSNC7AAMoX7q_-VjOM

Lowrie, T., Harris, D., Logan, T., & Hegarty, M. (2021). The impact of a spatial intervention program on students' spatial reasoning and mathematics performance. *Journal of Experimental Education*, *89*(2), 259–277. https://doi.org/10.1080/00220973.2019.1684869

Lowrie, T., & Logan, T. (2007). Using spatial skills to interpret maps: Problem solving in realistic contexts. *Australian Primary Mathematics Classroom, 12*(4), 14–19. https://search.informit.org/doi/epdf/10.3316/informit.136014411207891

Lowrie, T., & Logan, T. (2023). Spatial visualization supports students' math: Mechanisms for spatial transfer. *Journal of Intelligence*, *11*(6)127. 1–21. https://doi.org/10.3390/jintelligence11060127

Lowrie, T., Logan, T., Harris, D. & Hegarty, M. (2018). The impact of an intervention program on students' spatial reasoning: Student engagement through mathematics-enhanced learning activities. *Cognitive Research: Principles and Implications, 3*(1), 1–10. https://doi.org/10.1186/s41235-018-0147-y

Lowrie, T., Logan, T., & Hegarty, M. (2019). The influence of spatial visualization training on students' spatial reasoning and mathematics performance. *Journal of Cognition and Development*, *20*(5), 729–751. https://doi.org/10.1080/15248372.2019.1653298

Lowrie, T., Logan, T., & Ramful., A. (2016). Spatial reasoning influences students' performance on mathematics tasks. In White, B., Chinnappan, M. & Trenholm, S. (Eds.) *Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australasia*, (pp. 407–414). MERGA. https://merga.net.au/Public/Public/Publications/Annual_Conference_Proceedings/2016_MERGA_Conference_Proceedings.aspx

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal For Research in Mathematics Education, 21*(1), 16–32. https://doi.org/10.5951/jresematheduc.21.1.0016

Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education, 32*(3), 267–295. https://doi.org/10.2307/749828

Maeda, Y., & Yoon, S. Y. (2013). A meta-analysis on gender differences in mental rotation ability measured by the Purdue spatial visualization tests: Visualization of rotations (PSVT:R). *Educational Psychology Review, 25*(1), 69–94. https://doi.org/10.1007/s10648-012-9215-x

Maher, C. A., & Sigley, R. (2020). Task-based interviews in mathematics education. *Encyclopedia Of Mathematics Education*, 821–824. https://doi.org/10.1007/978-3-030-15789-0

Mamede, E., & Nunes, T. (2008). Building on children's informal knowledge in the teaching of fractions. In L. R. Wiest, & T. D. Lamberg (Eds.), *Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education and 30th North American Chapter of the International Group for the Psychology of Mathematics Education, 3* (pp. 345–352). PME. https://www.pmena.org/pmenaproceedings/PMENA%2033%202011%20Proceedings.pdf

Mamede, E., Nunes, T. & Bryant, P. (2005). *The equivalence and ordering of fractions in part-whole and quotient situations*. https://www.researchgate.net/publication/242687669_The_equivalence_and_ordering_of_fractions_in_part-whole_and_quotient_situations

Mamede, E. & Oliveira M. (2011). Issues on children's ideas of fractions when quotient interpretation is used. In Pytlak, M., Rowland, T., & Swoboda, E. (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education,* (pp. 1782 – 1791). http://erme.site/wp-content/uploads/2021/06/CERME7.pdf

Mainali, B. (2021). Representation in teaching and learning mathematics. *International Journal of Education in Mathematics, Science and Technology, 9*(1), 1–21. https://files.eric.ed.gov/fulltext/EJ1282034.pdf

Marston, J. (2010). Developing a framework for the selection of picture books to promote early mathematical development. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 383–390). MERGA. https://files.eric.ed.gov/fulltext/ED520914.pdf

Martens, S. E., Meeuwissen, S. N., Dolmans, D. H., Bovill, C., & Könings, K. D. (2019). Student participation in the design of learning and teaching: Disentangling the terminology and approaches. *Medical Teacher, 41*(10), 1203–1205. https://doi.org/10.1080/0142159X.2019.1615610

Matthews, P. G. & Chesney, D. L. (2015). Fractions as percepts? Exploring cross-format distance effects for fractional magnitudes. *Cognitive Psychology*, *78,* 28–56. https://doi.org/10.1016/j.cogpsych.2015.01.006

Matthews, P. G., & Ellis, A. B. (2018). Natural alternatives to natural number: The case of ratio. *Journal of Numerical Cognition, 4*(1), 19-58. https://doi.org/10.5964/jnc.v4i1.97

Matthews, P. G., & Lewis, M. R. (2017). Fractions we cannot ignore: The nonsymbolic ratio congruity effect. *Cognitive Science*, *41*(6), 1656–1674. https://doi.org/10.1111/cogs.12419

Matthews, P. G. & Ziols, R. (2019). What's perception got to do with it? Re-framing foundations for rational number concepts. In A. Norton & Alabali, M. (Eds.), *Constructing number* (pp. 213–235). Springer. https://doi.org/10.1007/978-3-030-00491-0_10

Mazzocco, M. M., Feigenson, L., & Halberda, J. (2011). Preschoolers' precision of the approximate number system predicts later school mathematics performance. *PLoS One, 6*(9), e23749. https://doi.org/10.1371/journal.pone.0023749

McCrink, K., & Spelke, E. S. (2010). Core multiplication in childhood. *Cognition, 116*(2), 204–216. https://doi.org/10.1016/j.cognition.2010.05.003

McCrink, K., & Wynn, K. (2007). Ratio abstraction by 6-month-old infants. *Psychological Science, 18*(8), 740–745. https://doi.org/10.1111/j.1467-9280.2007.01969.x

MacDonald, A. (2013). Researching with young children: Considering issues of ethics and engagement. *Contemporary Issues in Early Childhood*, *14*(3), 255–269. https://doi.org/10.2304/ciec.2013.14.3.2

MacDonald, A., & Lowrie, T. (2011). Developing measurement concepts within context: Children's representations of length. *Mathematics Education Research Journal*, *23*, 27–42. https://doi.org/10.1007/s13394-011-0002-7

McGaw, B., Louden, W. & Wyatt-Smith, C. (2020). *NAPLAN review: Final report*. Department of Education and Training (NSW). https://naplanreview.com.au/pdfs/2020_NAPLAN_review_final_report.pdf

McGee, M. G. (1979). Human spatial abilities: Psychometric studies and environmental, genetic, hormonal, and neurological influences. *Psychological Bulletin*, *86*(5), 889. https://doi.org/10.1037/0033-2909.86.5.889

McNeil, N. M. & Uttal, D. H. (2009). Rethinking the use of concrete materials in learning: Perspectives from development and education. *Child Development Perspectives*, *3*(3), 137–139. https://doi.org/10.1111/j.1750-8606.2009.00093.x

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. University of Chicago Press.

Middleton, J. A., Cai, J. & Hwang, S. (2015). Why mathematics education needs large-scale research. In J. A. Middleton, J. Cai, & S. Hwang (Eds,). *Large-scale studies in mathematics education*, (pp. 1–13). Springer. httsp://doi.org/10.1007/978-3-319-07716-1_1/

Mildenhall, P. M. (2013). Using semiotic resources when building images of the part-whole model of fractions. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Proceedings of Conference of the Mathematics Education Research Group of Australasia* (pp. 506–512). MERGA. https://files.eric.ed.gov/fulltext/ED572981.pdf

Miller, J. (2014). *Young indigenous students' experiences in mathematics: An exploration in pattern generalisation* [Doctoral Thesis]. https://doi.org/10.4226/66/5a975cba3baac

Misquitta, R. (2011). A review of the literature: Fraction instruction for struggling learners in mathematics. *Learning Disabilities Research & Practice, 26*(2), 109–119. https://doi.org/10.1111/j.1540-5826.2011.00330.x

Mitchell, A. & Horne, M. (2011). Listening to children's explanations of fraction pair tasks: When more than an answer and an initial explanation are needed. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices: Proceedings of the AAMT-MERGA Conference* (pp. 515–522). Mathematics Education Research Group of Australasia. https://www.merga.net.au/publications/counter.php?pub=pub_conf&id=1700

Mitchelmore, M. C., & White, P. (1995). Abstraction in mathematics: Conflict, resolution, and application. *Mathematics Education Research Journal, 7*(1), 50–68. https://doi.org/10.1007/BF03217275

Mix, K. S. & Cheng, Y. L. (2012). The relation between space and math: Developmental and educational implications. *Advances in Child Development and Behaviour*, *42*, 197–243. https://doi.org/10.1016/B978-0-12-394388-0.00006-X

Mix, K. S. & Levine, S. C. (2018). Part II commentary 2: Disparities and opportunities: Plotting a new course for research on spatial visualization and mathematics. In S. Mix & T. Battista (Eds.), *Visualizing mathematics: The role of spatial reasoning in mathematical thought* (pp. 347–353). Springer. https://doi.org/10.1007/978-3-319-98767-5

Mix, K. S., Levine, S. C., Cheng, Y. L., Young, C., Hambrick, D. Z., Ping, R. & Konstantopoulos, S. (2016). Separate but correlated: The latent structure of space and mathematics across development. *Journal of Experimental Psychology: General*, *145*(9), 1206–1227. https://doi.org/10.1037/xge0000182

Mix, K. S., Levine, S. C. & Huttenlocher, J. (1999). Early fraction calculation ability. *Developmental Psychology*, *35*(1), 164–174. https://doi.org/10.1037/0012-1649.35.1.164

Möhring, W., Frick, A. & Newcombe, N. S. (2018). Spatial scaling, proportional thinking, and numerical understanding in 5–to 7–year–old children. *Cognitive Development*, *45*, 57–67. https://doi.org/10.1016/j.cogdev.2017.12.001

Möhring, W., Libertus, M. E. & Bertin, E. (2012). Speed discrimination in 6-and 10-month-old infants follows Weber’s law. *Journal of Experimental Child Psychology*, *111*(3), 405–418. https://doi.org/10.1016/j.jecp.2011.11.002

Möhring, W., Newcombe, N. S. & Frick, A. (2014). Zooming in on spatial scaling: Preschool children and adults use mental transformations to scale spaces. *Developmental Psychology*, *50*(5), 1614–1619. https://doi.org/10.1037/a0035905

Möhring, W., Newcombe, N. S. & Frick, A. (2015). The relation between spatial thinking and proportional reasoning in pre-schoolers. *Journal of Experimental Child Psychology*, *132*, 213–220. https://doi.org/10.1016/j.jecp.2015.01.005

Möhring, W., Newcombe, N. S., Levine, S. C. & Frick, A. (2016). Spatial proportional reasoning is associated with formal knowledge about fractions. *Journal of Cognition and Development*, *17*(1), 67–84. https://doi.org/10.1080/15248372.2014.996289

Montessori, M. (1912). A critical consideration of the new pedagogy in its relation to modern science. In M. Montessori & A. E. George (Trans.), *The Montessori method: Scientific pedagogy as applied child education in 'The Children's Houses'*, with additions and revisions by the author (pp. 1–27). Frederick A Stokes Company. https://doi.org/10.1037/13054-001

Morsella, E., & Krauss, R. M. (2004). The role of gestures in spatial working memory and speech. *American Journal of Psychology*, *117*(3), 411–424. https://doi.org/10.2307/4149008

Moss, J. & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, *30*(2), 122–147. https://doi.org/10.2307/749607

Mulligan, J. (2015). Looking within and beyond the geometry curriculum: Connecting spatial reasoning to mathematics learning. *ZDM – Mathematics Education*, *47*, 511–517. https://doi.org/10.1007/s11858-015-0696-1

Mulligan, J., & Mitchelmore, M. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal*, *21*(2), 33–49. https://doi.org/10.1007/BF03217544

Mulligan, J. T., & Mitchelmore, M. C. (2013). Early awareness of mathematical pattern and structure. In L. D. English, & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning: advances in mathematics education*, (pp. 29–45). Springer. https://doi.org/10.1007/978-94-007-6440-8_3

Mulligan, J., & Mitchelmore, M. (2016). *Pattern and Structure Mathematics Awareness Program (PASMAP): Book One–Foundation and Year 1*. Australian Council for Educational Research.

Mulligan, J., Mitchelmore, M., & Prescott, A. E. (2005). Case studies of children's development of structure in early mathematics. In H. L. Chick, & J. L. Vincent, (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, (vol. 4, pp. 1–8.). https://opus.lib.uts.edu.au/bitstream/10453/7452/1/2005002178.pdf

Mulligan, J., Woolcott, G., Mitchelmore, M., Busatto, S., Lai, J., & Davis, B. (2020). Evaluating the impact of a Spatial Reasoning Mathematics Program (SRMP) intervention in the primary school. *Mathematics Education Research Journal*, *32*, 285–305. https://doi.org/10.1007/s13394-020-00324-z

Mulligan, J., Woolcott, G., Mitchelmore, M. & Davis, B. (2018). Connecting mathematics learning through spatial reasoning. *Mathematics Education Research Journal*, *30*(1), 77–87. https://doi.org/10.1007/s13394-017-0210-x

Munn, P. (1998). Symbolic function in pre-schoolers. In C. Donlan (Ed.), *The development of mathematical skills* (pp. 47–71). Psychology Press/Taylor & Francis (UK).

National Research Council. (2006). *Learning to think spatially*. The National Academy Press. https://www.nap.edu/read/11019/chapter/1

Newcombe, N. S. (2010). Picture this: Increasing math and science learning by improving spatial thinking. *American Educator*, *34*(2), 29–35. https://eric.ed.gov/?id=EJ889152

Newcombe, N. S. & Huttenlocher, J. (2003). *Making space: The development of spatial representation and reasoning*. MIT Press.

Ni, Y. & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27–52. https://doi.org/10.1207/s15326985ep4001_3

Ng, O. & Sinclair, N. (2013) Gestures and temporality: Children's use of gestures on spatial transformation tasks. In Lindmeier, A. M. & Heinze, A. (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 3, pp. 361–368). https://www.igpme.org/wp-content/uploads/2019/05/PME37-2013-Kiel.zip

Noelting, G. (1980). The development of proportional reasoning and the ratio concept part I–differentiation of stages. *Educational Studies in Mathematics, 11*(2), 217–253. https://www.jstor.org/stable/3481919

Norton, A., & Boyce, S. (2013). A cognitive core for common state standards. *Journal of Mathematical Behavior, 32*(2), 266–279. https://doi.org/10.1016/j.jmathb.2013.01.001

Norton, A. & Hackenberg, A. J. (2010). Continuing research on students' fraction schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 341–352). Springer.

Norton, A., & Wilkins, J. L. (2012). The splitting group. *Journal for Research in Mathematics Education, 43*(5), 557-583. https://doi.org/10.5951/jresematheduc.43.5.0557

Norton, A., & Wilkins, J. L. (2013). Supporting students' constructions of the splitting operation. *Cognition and Instruction, 31*(1), 2-28. https://doi.org/10.1080/07370008.2012.742085

Nuerk, H. C. & Willmes, K. (2005). On the magnitude representations of two-digit numbers. *Psychology Science, 47*(1), 52–72. https://pdfs.semanticscholar.org/b1ae/00533a2b540b1b2501bb6b08011cd9ff96fd.pdf

Nunes, T., & Bryant, P. (1995). Do problem situations influence children's understanding of the commutativity of multiplication? *Mathematical Cognition, 1*(2), 245–260.

Nunes, T. & Bryant, P. (1996). *Children doing mathematics*. Wiley-Blackwell.

Nunes, T. & Bryant, P. (2007). *Key understandings in mathematics learning. Paper 3: Understanding rational numbers and intensive quantities*. Nuffield Foundation. https://www.nuffieldfoundation.org/wp-content/uploads/2020/03/P3.pdf

Obersteiner, A., Dresler, T., Bieck, S. M. & Moeller, K. (2019). Understanding fractions: Integrating results from mathematics education, cognitive psychology, and neuroscience. In A. Norton & Alabali, M. (Eds.), *Constructing number* (pp. 135–162). Springer. https://doi.org/10.1007/978-3-030-00491-0_8

Ojose, B. & Sexton, L. (2009). The effect of manipulative materials on mathematics achievement of first grade students. *The Mathematics Educator*, *12*(1), 3–14. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.1038.5734&rep=rep1&type=pdf

Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning*, *1*(4), 279–314. https://doi.org/10.1207/s15327833mtl0104_2

Olive, J., & Lobato, J. (2008). The learning of rational number concepts using technology. In M. K. Heid & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Research syntheses, 1*, 1–53. Information Age Publishing.

Ontario Ministry of Education. (2014). *Paying attention to spatial reasoning*. Queen’s Printer for Ontario. http://www.edu.gov.on.ca/eng/literacynumeracy/lnspayingattention.pdf

Onwuegbuzie, A. J., & Leech, N. L. (2007). Validity and qualitative research: An oxymoron?. *Quality & Quantity, 41*, 233–249. https://doi.org/10.1007/s11135-006-9000-3

Orlikowski, W. J., & Baroudi, J. J. (1991). Studying information technology in organizations: Research approaches and assumptions. *Information Systems Research*, *2*(1), 1–28. https://www.jstor.org/stable/23010611

Patahuddin, S., Logan, T., & Ramful, A. (2018). Characteristics of spatial visualisation: Perspectives from area of composite shapes. In J., Hunter, P., Perger, & L. Darragh, (Eds.). *Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 623–630). MERGA. https://files.eric.ed.gov/fulltext/ED592438.pdf

Pape, S. J., & Tchoshanov, M. A. (2001). The role of representation (s) in developing mathematical understanding. *Theory Into Practice, 40*(2), 118–127. https://doi.org/10.1207/s15430421tip4002_6

Papic, M., Mulligan, J., & Mitchelmore, M. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42*(3), 237–269. https://doi.org/10.5951/jresematheduc.42.3.0237

Payne, J. N. (1976). Review of research on fractions. In R. Lesh (Ed.), *Number and Measurement* (pp. 145–188). University of Georgia.

Pearn, C. & Stephens, M. (2007). Whole number knowledge and number lines help develop fraction concepts. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia,* (Vol. 2, pp. 601–610). MERGA. https://merga.net.au/Public/Publications/Annual_Conference_Proceedings/2007_MERGA_CP.aspx

Pearn, C. & Stephens, M. (2016). Competence with fractions in fifth or sixth grade as a unique predictor of algebraic thinking*?* In B. White, M. Chinnappan, & S. Trenholm, (Eds.), *Proceedings of the 39th Annual Conference of the Mathematics Education Research Group of Australasia*. 519 – 526. https://eric.ed.gov/?id=ED572362

Pedersen, P.L., Bjerre, M. (2021). Two conceptions of fraction equivalence. *Education Studies in Mathematics, 107*, 135–157. https://doi.org/10.1007/s10649-021-10030-7

Pepper, K. L. (1991). Preschoolers' knowledge of counting and sharing in discrete quantity settings. In R. Hunting & G. Davis (Eds.), *Early fraction learning* (pp. 103–129). Springer. https://doi.org/10.1007/978-1-4612-3194-3_7

Piaget, J. (1936). *Origins of intelligence in the child*. Routledge & Kegan Paul.

Piaget, J. (1964). Cognitive development in children. *Journal of Research in Science Teaching*, *2*(2), 176–186. https://doi.org/10.1002/tea.3660020306

Piaget, J. & Inhelder, B. (1967). The coordination of perspectives. In *The child's conception of space* (pp. 209–246). Psychology Press.

Pianta, R. & Hamre, B. (2009). Conceptualization, measurement, and improvement of classroom processes: Standardized observation can leverage capacity. *Educational Researcher, 38*, 109–119. https://doi.org/10.3102/0013189X09332374.

Pitkethly, A. & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics, 30*(1), 5–38. https://doi.org/10.1007/BF00163751

Posner, G. J., Strike, K. A., Hewson, P. W., & Gertzog, W. A. (1982). Accommodation of scientific conception: Towards a theory of conceptual change, *Science Education*, *66,* 211-227. https://staff.science.uva.nl/e.joling/vakdidactiek/documenten/posner.pdf

Pothier, Y. & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education, 14*(5), 307–317. https://doi.org/10.2307/748675

Pothier, Y. & Sawada, D. (1990). Partitioning: An approach to fractions. *The Arithmetic Teacher, 38*(4), 12–16. https://search.proquest.com/docview/208771953?pq-origsite=gscholar

Powell, A. B. (2023). Enhancing students' fraction magnitude knowledge: A study with students in early elementary education. *The Journal of Mathematical Behaviour, 70*, 101042. https://doi.org/10.1016/j.jmathb.2023.101042

Praetorius, A. & Charalambous, C. Y. (2018). Classroom observation frameworks for studying instructional quality: Looking back and looking forward. *ZDM – Mathematics Education, 50*, 535–553. https://doi.org/10.1007/s11858-018-0946-0

Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes: An overview on achievements and challenges. *ZDM – Mathematics Education, 47*, 877–891. https://doi.org/10.1007/s11858-015-0722-3

Presmeg, N. C. (1986). Visualisation in high school mathematics. *For the Learning of Mathematics*, *6*(3), 42–46. https://flm-journal.org/Articles/1917B083BE4534511A32616EED75A8.pdf

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 205–235). Brill Sense.

Queensland Studies Authority. (2013). *Queensland students' understanding of fractions: Evidence from the NAPLAN test results.* https://www.qcaa.qld.edu.au/downloads/publications/report_students_understanding_fractions.pdf.

Rapp, M., Bassok, M., DeWolf, M., & Holyoak, K. J. (2015). Modeling discrete and continuous entities with fractions and decimals. *Journal of Experimental Psychology: Applied, 21*(1), 47– 56. http://dx.doi.org/10.1037/xap0000036

Reeder, S., & Utley, J. (2017). What is a fraction? Developing fraction understanding in prospective elementary teachers. *School Science and Mathematics*, *117*(7–8), 307–316. https://doi.org/10.1111/ssm.12248

Resnick, L. B. (1983). A developmental theory of number understanding. In H. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 109–151). Academic Press.

Resnick, L. B., & Singer, J. A. (1993). Protoquantitative origins of ratio reasoning. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 107–130). Lawrence Erlbaum Associates, Inc.

Richard-Bollans, A., Gómez Álvarez, L., Bennett, B. & Cohn, A. G. (2019). Investigating the dimensions of spatial language. In K. Stock, C. B. Jones & T. Tenbrink (Eds.), *Proceedings speaking of location 2019*, 47–56. http://eprints.whiterose.ac.uk/148335/8/Investigating_the_Dimensions_of_Spatial_Language.pdf

Robutti, O., Sabena, C., Krause, C., Soldano, C., & Arzarello, F. (2022). Gestures in mathematics thinking and learning. In M. Danesi (Ed.) *Handbook of cognitive mathematics* (pp. 1–42). Springer International Publishing. https://dx.doi.org/10.1007/978-3-031-03945-4_8

Rogers, A. (2014). *Investigating whole number place value in Years 3–6: creating an evidence-based developmental progression* [Doctoral Thesis]. https://researchrepository.rmit.edu.au/esploro/outputs/doctoral/Investigating-whole-number-place-value-in/9921864031901341#file-0

Saenz-Ludlow, A. (1994). Michael’s fraction schemes. *Journal for Research in Mathematics Education*, *25*(1), 50–85. https://doi.org/10.2307/749292

Saenz-Ludlow, A. (1995). Ann's fraction schemes. *Educational Studies in Mathematics, 28*(2), 101–132. https://doi.org/10.1007/BF01295789

Sarama, J., & Clements, D. H. (2009). “Concrete” computer manipulatives in mathematics education. *Child Development Perspectives, 3*(3), 145–150. https://doi.org/10.1111/j.1750-8606.2009.00095.x

Sarty, G. E. (2020). *Introduction to Applied Statistics for Psychology Students*. University of Saskatchewan Open Press.

Sasanguie, D., Göbel, S. M., Moll, K., Smets, K. & Reynvoet, B. (2013). Approximate number sense, symbolic number processing, or number–space mappings: What underlies mathematics achievement? *Journal of Experimental Child Psychology*, *114*(3), 418–431. https://doi.org/10.1016/j.jecp.2012.10.012

Scharp, K. M., & Sanders, M. L. (2019). What is a theme? Teaching thematic analysis in qualitative communication research methods. *Communication Teacher, 33*(2), 117–121. https://doi.org/10.1080/17404622.2018.1536794

Schlottmann, A. (2001). Children’s probability intuitions: Understanding the expected value of complex gambles. *Child Development*, *72*(1), 103–122. https://doi.org/10.1111/1467-8624.00268

Schneider, M. & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance*, *36*(5), 1227–1238. https://doi.org/10.1037/a0018170

Seah, R. T. K., & Horne, M. (2020). The influence of spatial reasoning on analysing about measurement situations. *Mathematics Education Research Journal*, *32*, 365–386. https://doi.org/10.1007/s13394-020-00327-w.

Seethaler, P. M., Fuchs, L. S., Star, J. R. & Bryant, J. (2011). The cognitive predictors of computational skill with whole versus rational numbers: An exploratory study. *Learning and Individual Differences*, *21*(5), 536–542. https://doi.org/10.1016/j.lindif.2011.05.002

Sembiring, R. K., Hadi, S., & Dolk, M. (2008). Reforming mathematics learning in Indonesian classrooms through RME. *ZDM – Mathematics Education, 40*, 927–939. https://doi.org/10.1007/s11858-008-0125-9

Schwandt, T. A. (1998). The Interpretive Review of Educational Matters: Is There Any Other Kind? *Review of Educational Research*, *68*(4), 409–412. https://doi.org/10.3102/00346543068004409

Siebert, D., & Gaskin, N. (2006). Creating, naming, and justifying fractions. *Teaching Children Mathematics*, *12*(8), 394–400. https://doi.org/10.2307/41198803

Siegler, R. S. (2016). Magnitude knowledge: The common core of numerical development. *Developmental Science, 19*(3), 341–361. https://doi.org/10.1111/desc.12395

Siegler, R. S., & Braithwaite, D. W. (2017). Numerical development. *Annual Review of Psychology*, *68,* 187-213. https://doi.org/10.1146/annurev-psych-010416-044101

Siegler, R. S., Duncan, G. J., Davis–Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I. & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691–697. http://doi.org/10.1177/0956797612440101

Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Sciences, 17*(1), 13–19. https://doi.org/10.1016/j.tics.2012.11.004

Siegler, R. S. & Lortie-Forgues, H. (2014). An integrative theory of numerical development. *Child Development Perspectives*, *8*(3), 144–150. https://doi.org/10.1111/cdep.12077

Siegler, R. S. & Opfer, J. E. (2003). The development of numerical estimation: Evidence for multiple representations of numerical quantity. *Psychological Science*, *14*(3), 237–250. https://doi.org/10.1111/1467-9280.02438

Siegler, R. S. & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology*, *49*(10), 1994–2004. https://doi.org/10.1037/a0031200

Siegler, R. S., Thompson, C. A., & Opfer, J. E. (2009). The logarithmic–to–linear shift: One learning sequence, many tasks, many time scales. *Mind, Brain, and Education*, *3*(3), 143–150. https://doi.org/10.1111/j.1751-228X.2009.01064.x

Siegler, R. S., Thompson, C. A. & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–296. https://doi.org/10.1016/j.cogpsych.2011.03.001

Siemon, D. (2000). *Multiplicative thinking: A professional development seminar.* [PowerPoint Slides]. https://makeitcount.aamt.edu.au/content/download/33536/473777/version/1/file/PD+Multiplicative+Thinking.pdf

Siemon, D. (2003). Partitioning: the missing link in building fraction knowledge and confidence. *Australian Mathematics Teacher*, *59*(3), 22–24. https://search.informit.org/doi/pdf/10.3316/aeipt.130543?casa_token=L7rq6u-UB_EAAAAA:pFIBrVTWKiFzVWWYX9L4PFMOLkJrYfYsq4Z26Hpvqpg5SdYaZ0T1PBcTnGqnkzOvyPSaCSpiokl9sxsF

Siemon, D. (2006). *Assessment for Common Misunderstandings Materials.* Victorian Department of Education and Early Childhood Development. http://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/ Pages/misunderstandings.aspx

Siemon, D. (2010). A note on ‘common misunderstandings’. *Prime Number*, *25*(1), 3–5. https://search.informit.com.au/documentSummary;dn=736996579514078;res=IELHSS

Siemon, D. (2013). Launching mathematical futures: The key role of multiplicative thinking. In S. Herbert, J. Tillyer & T. Spencer (Eds.), *Proceedings of the 24th Biennial Conference of The Australian Association of Mathematics Teachers Inc.* (pp.36–52). AAMT. https://primarystandards.aamt.edu.au/content/download/38258/547092/file/maths_launching_futures.pdf

Siemon, D. (2016). Addressing the STEM challenge through targeted teaching : What’s the evidence? *ACER Research Conference 2016 – Improving STEM Learning : What will it take?* Retrieved from https://research.acer.edu.au/research_conference/RC2016/9august/3

Siemon, D. (2019). Knowing and building on what students know: The case of multiplicative thinking. In D. Siemon, T. Barkatsas, R. Seah (Eds.), *Researching and using progressions (trajectories) in mathematics education*. Brill. https://doi.org/10.1163/9789004396449_001

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R. & Warren, E. (2015). *Teaching mathematics: Foundations to middle years* (3rd ed.). Oxford University Press.

Siemon, D., Bleckly, J. & Neal, D. (2012). Working with the big ideas in number and the Australian Curriculum: Mathematics. In B. Atweh, M. Goos, R. Jorgensen & D. Siemon, (Eds.) *Engaging the Australian national curriculum: Mathematics – perspectives from the field* (pp. 19–45). Mathematics Education Research Group of Australasia.

Siemon, D., Breed, M., Dole, S., Izard, J. & Virgona, J. (2006). *Scaffolding numeracy in the middle years – Project findings, materials, and resources*. Final Report submitted to Victorian Department of Education and Training and the Tasmanian Department of Education. CD–ROM subsequently published electronically: http://www.education.vic.gov.au/studentlearning/teachingresources/maths/snmy/snmyassessmult.htm

Siemon, D., Horne, M., Clements, D., Confrey, J., Maloney, A., Sarama, J., ... & Watson, A. (2017). Researching and using learning progressions (trajectories) in mathematics education. In In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.) *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (pp. 109–136). PME. https://www.igpme.org/wp-content/uploads/2019/05/PME41-2017-Singapore.zip

Siemon, D., Virgona, J. & Corneille, K. (2001). *The middle years numeracy research project: 5–9. Final report*. RMIT University. https://www.education.vic.gov.au/Documents/school/teachers/teachingresources/discipline/maths/mynumfreport.pdf

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114–145. https://www.jstor.org/stable/749205

Simon, M. & Dougherty, B. (2014). Elkonin and Davydov curriculum in mathematics education. In S. Lermean (Ed.), *Encyclopedia of mathematics education* (pp. 204–207). Springer.

Simon, M. A., Placa, N., Avitzur, A. & Kara, M. (2018). Promoting a concept of fraction–as–measure: A study of the Learning Through Activity research program. *The Journal of Mathematical Behaviour*, *52*, 122–133. https://doi.org/10.1016/j.jmathb.2018.03.004

Sinclair, N. & Bruce, C. D. (2014). *Spatial reasoning for young learners*. In Liljedahl, P., Nicol, C., Oesterle, S., & Allan, D. (Eds.). *Proceedings of the Joint Meeting of Psychology of Mathematics Education 38 and Psychology of Mathematics Education – North American Chapter 36*, (Vol. 1) https://www.academia.edu/download/40348841/Research_Forum_-_Spatial_Reasoning_in_Young_Children_p.25-26.pdf

Singer-Freeman, K. E. & Goswami, U. (2001). Does half a pizza equal half a box of chocolates?: Proportional matching in an analogy task. *Cognitive Development, 16*(3), 811–829. https://doi.org/10.1016/S0885-2014(01)00066-1

Singer, J. A., & Resnick, L. B. (1992). Representations of proportional relationships: Are children part–part or part–whole reasoners*?. Educational Studies in Mathematics, 23*(3), 231–246. https://doi.org/10.1007/BF02309531

Smith, J. K. (1992). Interpretive inquiry: A practical and moral activity. *Theory Into Practice*, *31*(2), 100–106. https://doi.org/10.1080/00405849209543530

Sophian, C., Garyantes, D. & Chang, C. (1997). When three is less than two: Early developments in children's understanding of fractional quantities. *Developmental Psychology*, *33*(5), 731–744. https://doi.org/10.1037/0012-1649.33.5.731

Sophian, C., & Madrid, S. (2003). Young children's reasoning about many–to–one correspondences. *Child Development*, *74*(5), 1418–1432. https://doi.org/10.1111/1467-8624.00615

Sorby, S. A. (1999). Developing 3-D spatial visualization skills. *Engineering Graphics Design Journal*, *63*(2), 21–32.http://diggingdeeper.pbworks.com/f/Developing+Spatial+Skills.pdf

Spencer, P. (2017). *Teaching and learning spatial thinking with young students: The use and influence of external representations* [Doctoral Thesis]. https://doi.org/10.26199/5b84dd08c927f

Spinillo, A. G. & Bryant, P. (1991). Children's proportional judgments: The importance of 'half'. *Child Development*, *62*(3), 427–440. https://doi.org/10.1111/j.1467-8624.1991.tb01542.x

Stafylidou, S. & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*(5), 503–518. https://doi.org/10.1016/j.learninstruc.2004.06.015

Starr, A. & Brannon, E. M. (2015). Evidence against continuous variables driving numerical discrimination in infancy. *Frontiers in Psychology*, *6*, 923. https://doi.org/10.3389/fpsyg.2015.00923

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 119–140). National Council of Teachers of Mathematics.

Steffe, L. P. (2001). A new hypothesis concerning children's fractional knowledge. *The Journal of Mathematical Behaviour*, *20*(3), 267–307. https://doi.org/10.1016/S0732-3123(02)00075-5

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning, 6*(2), 129–162. https://doi.org/10.1207/s15327833mtl0602_4

Steffe, L. P. & Olive, J. (2010). *Children’s fractional knowledge*. Springer. https://doi.org/10.1007/978-1-4419-0591-8

Steffe, L. P., Von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children’s counting types: Philosophy, theory, and application.* Praeger Scientific.

Stohl, H. (2005). Probability in teacher education and development. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 345–366). Kluwer Academic Publishers. https://link.springer.com/content/pdf/10.1007/0-387-24530-8_15.pdf

Stevens, A. L., Wilkins, J. L. M., Lovin, L. H., Siegfried, J., Norton, A., & Busi, R. (2020). Promoting sophisticated fraction constructs through instructional changes in a mathematics course for PreK–8 prospective teachers. *The Journal of Mathematics Teacher Education, 23,* 153–181. https://doi.org/10.1007/s10857-018-9415-5

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research.* Kluwer Academic Publishers. https://doi.org/10.1007/978-94-011-3168-1

Streefland, L. (1993a). The design of a mathematics course. A theoretical reflection. *Educational Studies in Mathematics, 25*(1–2), 109-135. https://doi.org/10.1007/BF01274105

Streefland, L. (1993b). Fractions: A realistic approach. In T. P. Carpenter, E. A. Fennema & T. A. Romberg (Eds.), *Rational numbers an integration of research* (pp. 289–325). Lawrence Erlbaum.

Strike, K. A., & Posner, G. J. (1992). A revisionist theory of conceptual change. In R. A. Duschl & R. J. Hamilton (Eds.), *Philosophy of science, cognitive psychology, and educational theory and practice, 176,* 147–178. New York Press.

Strauss, A. L. (1987). *Qualitative Analysis for Social Scientists.* Cambridge University Press.

Sullivan, P., Askew, M., Cheeseman, J., Clarke, D., Mornane, A., Roche, A., & Walker, N. (2015). Supporting teachers in structuring mathematics lessons involving challenging tasks. *The Journal of Mathematics Teacher Education, 18*(2), 123–140. https://doi.org/10.1007/s10857-014-9279-2

Surahmi, E., Budayasa, I. K., & Ekawati, R. (2018). The Role of Gesture to Mathematics' Fraction Base Concept Instruction, *Journal of Physics: Conference Series* (Vol. 1108, pp. 012075). IOP Publishing. https://doi.org/ 10.1088/1742-6596/1108/1/012075

Swart, M. I., Friedman, B., Kornkasem, S., Hollenburg, S., Lowes, S., Black, J. B., Vitale J. M., Sheppard S. & Nankin, F. (2014). *Mobile movement mathematics: Exploring the gestures students make while explaining fractions*. In AERA National Conference, Philadelphia, PA.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory-based instruction: Toward a theory of teaching. *Educational Researcher, 41*(5), 147–156. http://www.jstor.org/stable/23254092

Tahta, D. (1989). Is there a geometric imperative? *Mathematics Teaching, 129,* 20–29.

Takeuchi, M. A., & Dadkhahfard, S. (2019). Rethinking bodies of learners through STEM education. In P. Sengupta., M-C. Shanahan., & B. Kim (Eds.), *Critical, transdisciplinary, and embodied approaches in STEM education* (pp. 199–216). Springer. https://doi.org/ 10.1007/978-3-030-29489-2_11

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169. https://doi.org/10.1007/BF00305619

Thomas, N. (2004). The development of structure in the number system. In M. J. Hoines & A. B. Fuglestad (Eds.), *28th Conference of the International Group for the Psychology of Mathematics Education*, (pp. 305–312). Bergen University College Press. https://www.igpme.org/wp-content/uploads/2019/05/PME28-2004-Bergen.zip

Thompson, P. W. & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *A research companion to the principles and standards for school mathematics* (pp. 95–114). National Council of Teachers of Mathematics. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.630.5632&rep=rep1&type=pdf

Thomson, S. (2020). Exploring Australian students' strengths and weaknesses in maths and science. *Teacher: Evidence+Insight+Action.* https://www.teachermagazine.com/au_en/articles/exploring-australian-students-strengths-and-weaknesses-in-maths-and-science

Thomson, S., De Bortoli, L. & Underwood, C. (2017b). *PISA 2015 Australia student and achievement data [SPSS]* [Dataset]. Australian Council for Educational Research (ACER). https://research.acer.edu.au/ozpisa/26

Thomson, S., Wernert, N., O'Grady, E. & Rodrigues, S. (2017a). *TIMSS 2015: Reporting Australia's results*. Australian Council for Educational Research (ACER). https://research.acer.edu.au/timss_2015/2/

Thomson, S., Wernert, N., Rodrigues, S. & O'Grady, E. (2020). *TIMSS 2019 Australia. Volume I: Student performance*. Australian Council for Educational Research (ACER). https://doi.org/10.37517/978-1-74286-614-7

Tobin, G. A., & Begley, C. M. (2004). Methodological rigour within a qualitative framework. *Journal of Advanced Nursing, 48*(4), 388–396. https://onlinelibrary.wiley.com/doi/pdf/10.1111/j.1365-2648.2004.03207.x

Trafton, J. G., Trickett, S. B., Stitzlein, C. A., Saner, L., Schunn, C. D., & Kirschenbaum, S. S. (2006). The relationship between spatial transformations and iconic gestures. *Spatial Cognition and Computation, 6*(1), 1–29. https://doi.org/10.1207/s15427633scc0601_1

Treagust, D. F., & Duit, R. (2008). Conceptual change: A discussion of theoretical, methodological, and practical challenges for science education. *Cultural Studies of Science Education, 3,* 297–328. http://dx.doi.org/10.1007/s11422-008-9090-4

Trochim, W. M. K. (2006). The qualitative debate. *Research Methods Knowledge Base.* http://www.socialresearchmethods.net/kb/qualmeth.php

Tsay, J., & Hauk, S. (2009). *Prospective Elementary Teachers' Multiplication Schema for Fractions*. Online publication, 1–16. http://bookwalk.net/papers/TsayHauk2009PST_LONG.pdf.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education*, *30*(4), 390–416. https://www.jstor.org/stable/pdf/749707.pdf?refreqid=excelsior%3A72e664f6d12dfe48e77ab74c7ebb7118

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: Participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics, 66*(3), 273–291. http://www.jstor.org/stable/27822706

Tzur, R. (2019). Developing fractions as multiplicative relations: A model of cognitive reorganization. In A. Norton & Alabali, M. (Eds.), *Constructing number* (pp. 163–191). Springer. https://doi.org/10.1007/978-3-030-00491-0_8

Ubah, I. J. A. & Bansilal, S. (2018). Pre-service primary Mathematics teachers' understanding of fractions: An action-process-object-schema perspective. *South African Journal of Childhood Education*, *8*(2), 1–12. https://dx.doi.org/10.4102/sajce.v8i2.539

University of Cambridge (1997–2003). *Millennium mathematics project*, Nrich. https://nrich.maths.org/frontpage

Usiskin, Z. P. (2007). Some thoughts about fractions. *Mathematics Teaching in the Middle School, 12*(7), 370–373. http://www.jstor.org/stable/41182432

Uttal, D. H. & Cohen, C. A. (2012). Spatial thinking and STEM education: When, why, and how? *Psychology of Learning and Motivation, 57*, 147–181. https://doi.org/10.1016/B978-0-12-394293-7.00004-2

Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C. & Newcombe, N. S. (2013). The malleability of spatial skills: A meta-analysis of training studies. *Psychological Bulletin*, *139*(2), 352–402. https://doi.org/10.1037/a0028446

Vamvakoussi, X., Christou, K. P., & Vosniadou, S. (2018). Bridging psychological and educational research on rational number knowledge. *Journal of Numerical Cognition, 4*(1), 84–106. https://doi.org/10.5964/jnc.v4i1.82

Vamvakoussi, X. & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, *14*(5), 453–467. https://doi.org/10.1016/j.learninstruc.2004.06.013

van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*, 521–534. Springer. https://doi.org/10.1007/978-94-007-4978-8

van den Heuvel-Panhuizen, M., Van Den Boogaard, S., & Doig, B. (2009). Picture books stimulate the learning of mathematics. *Australasian Journal of Early Childhood, 34*(3), 30–39. https://doi.org/10.1177/183693910903400305

Vanluydt, E., Degrande, T., Verschaffel, L. & Van Dooren, W. (2019). Early stages of proportional reasoning: a cross-sectional study with 5–to 9–year-olds. *European Journal of Psychology of Education*, *35*, 529–547. https://doi.org/10.1007/s10212-019-00434-8

Van Nes, F. & Van Eerde, D. (2010). Spatial structuring and the development of number sense: A case study of young children working with blocks. *The Journal of Mathematical Behavior*, *29*(3), 145–159. https://doi.org/10.1016/j.jmathb.2010.08.001

Vasilyeva, M. & Huttenlocher, J. (2004). Early development of scaling ability. *Developmental Psychology*, *40*(5), 682–690. https://doi.org/10.1037/0012-1649.40.5.682

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141–161). National Council of Teachers of Mathematics.

Von Glasersfeld, E., & Steffe, L. P. (1991). Conceptual models in educational research and practice. *Journal of Educational Thought (JET), 25*(2), 91–103. https://www.jstor.org/stable/23767267

Vukovic, R. K., Fuchs, L. S., Geary, D. C., Jordan, N. C., Gersten, R. & Siegler, R. S. (2014). Sources of individual differences in children's understanding of fractions. *Child Development*, *85*(4), 1461–1476. https://doi.org/10.1111/cdev.12218

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Harvard University Press.

Wai, J., Lubinski, D. & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology*, *101*(4), 817–835. https://doi.org/10.1037/a0016127

Walsham, G. (1995). Interpretive case studies in IS research: nature and method. European *Journal of Information Systems, 4*(2), 74–81. https://doi.org/10.1057/ejis.1995.9

Wang, Y. & Siegler, R. S. (2013). Representations of and translation between common fractions and decimal fractions. *Chinese Science Bulletin*, *58*(36), 4630–4640. https://doi.org/10.1007/s11434-013-6035-4

Wang, Y., & Siegler, R. S. (2023). Developmental trajectories of numerical magnitude representations of fractions and decimals. *Developmental Psychology*, *59*(5), 874–885. https://doi.org/10.1037/dev0001547

Watanabe, T. (2002). Representations in teaching and learning fractions. *Teaching Children Mathematics*, *8*(8), 457–463. https://link.gale.com/apps/doc/A84903077/AONE?u=unisa&sid=googleScholar&xid=042a7b5a

Way, J. (2011). Developing fraction sense using digital learning objects. *Fractions: Teaching for understanding*, 153–166. Australian Association of Mathematics Teachers. https://primarystandards.aamt.edu.au/content/download/19932/273057/file/tdt_F_way1.pdf

Way, J., Bobis, J. & Anderson, J. (2015). Teacher representations of fractions as a key to developing their conceptual understanding. In K. Beswick, T. Muir & J. Wells (Eds.), *Proceedings of 39th Psychology of Mathematics Education Conference* (vol. 4, pp. 281–288). PME. https://www.igpme.org/wp-content/uploads/2019/05/PME39-2015-Hobart.zip

Webster, V. (2020). Preservice teachers' strategies for interpreting fractions represented in discrete and continuous models. *Transformations: Research papers.* https://digitalcommons.cortland.edu/cgi/viewcontent.cgi?article=1005&context=programs

Wilkie, K.J., Roche, A. (2022). Primary teachers' preferred fraction models and manipulatives for solving fraction tasks and for teaching. *Journal of Mathematics Teacher Education*, 1–31. https://doi.org/10.1007/s10857-022-09542-7

Wilkins, J. L. & Norton, A. (2018). Learning progression toward a measurement concept of fractions. *International Journal of STEM Education, 5*(1), 27. https://doi.org/10.1186/s40594-018-0119-2

Wilson, P. H., Myers, M., Edgington, C., & Confrey, J. (2012). Fair shares, matey, or walk the plank. *Teaching Children Mathematics, 18*(8), 482–489. https://www.jstor.org/stable/10.5951/teacchilmath.18.8.0482

Whiteley, W., Sinclair, N., & Davis B. (2015). What is spatial reasoning?, In B Davis, & The Spatial Reasoning Group (Eds.), *Spatial reasoning in the early years: Principles, assertions, and speculations.* Routledge.

Woleck, K. R. (2001). Listen to their pictures: An investigation of children's mathematical drawings. In A. Cuoco & F. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 215–227). National Council of Teachers of Mathematics.

Wong, M. & Evans, D. (2007). Students' conceptual understanding of equivalent fractions. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 824–833). MERGA. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.607.7837&rep=rep1&type=pdf

Worthington, M., & Carruthers, E. (2003). *Children's mathematics: Making marks, making meaning*. Sage Publications.

Wright, S. (2007). Young children's meaning-making through drawing and 'telling'. Analogies to filmic textual features. *Australasian Journal of Early Childhood, 32*(4), 37–48. https://doi.org/10.1177/183693910703200408

Xie, J. & Masingila, J. O. (2017). Examining interactions between problem posing and problem solving with prospective primary teachers: A case of using fractions. *Educational Studies in Mathematics, 96*(1), 101–118. https://doi.org/10.1007/s10649-017-9760-9

Xu, F. & Spelke, E. S. (2000). Large number discrimination in 6-month-old infants. *Cognition, 74*(1), 1–11. https://doi.org/10.1016/S0010-0277(99)00066-9

Ye, A., Resnick, I., Hansen, N., Rodrigues, J., Rinne, L. & Jordan, N. C. (2016). Pathways to fraction learning: Numerical abilities mediate the relation between early cognitive competencies and later fraction knowledge. *Journal of Experimental Child Psychology, 152*, 242–263. https://doi.org/10.1016/j.jecp.2016.08.001

Yearley, S., & Bruce, C. D. (2014). A Canadian effort to address fractions teaching and learning challenges. *Australian Primary Mathematics Classroom, 19*(4), 34–39. https://search.informit.org/doi/pdf/10.3316/informit.911385706321282

Young, C. J., Levine, S. C., & Mix, K. S. (2018). The connection between spatial and mathematical ability across development. *Frontiers in Psychology, 9*, 755. 2–7. https://doi.org/10.3389/fpsyg.2018.00755

Young–Loveridge, J. (2002). Early childhood numeracy: Building an understanding of part–whole relationships. *Australasian Journal of Early Childhood, 27*(4), 36–40. https://doi.org/10.1177/183693910202700408

Zhang, Q. (2016). Teaching of fraction in elementary schools based on understanding of different sub-constructs. In G. Kaiser (Ed.), *Proceedings of the 13th International Congress on Mathematical Education* (pp. 51–54). Springer. https://link.springer.com/book/10.1007/978-3-319-62597-3

Zhang, X., Clements, M. A. & Ellerton, N. E. (2015). Engaging students with multiple models of fractions. *Teaching Children Mathematics*, *22*(3), 138–147. https://doi.org/10.5951/teacchilmath.22.3.0138

Appendices

Appendix A: Cycle One Pilot Tasks

<table>
<tr><th colspan="5">Pilot Task 1: Sharing Cookies</th></tr>
<tr><th colspan="2">Rational Number Foci</th><th>Spatial Reasoning Foci</th><th colspan="2">Children trialling the task</th></tr>
<tr><th>Fraction as Operator</th><th>Fraction as Measure</th><th>Construct</th><th>Group 1</th><th>Group 2</th></tr>
<tr>
<td rowspan="2">Fair Share
The creation of equal size shares (of discrete collections or continuous wholes) where the shares created exhaust the whole. Naming fair shares of collections, including counting and relational naming (naming one share in relation to the whole collection or single whole).

Doubling and Halving
Derived from splitting (2-split) which is founded on repeated doubling and halving (splits of splits), perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015).

Partitive division/ recursive multiplication
The introduction to division and multiplication are viewed as inverse operations

1-nth-of……
The relationship between “1/nth of” in naming fair shares and identifying the referent units (greater or less than 1) for the fair shares resulting from equipartitioning (Confrey & Maloney, 2010, p. 973).</td>
<td rowspan="2">Many-as-one
Many-as-one is a group of m objects, where the quotient represents the extensive quantity that one sharer receives (Confrey, 2012)

Measure
Directly related to fair share, in that when fair shares are created, these shares represent a quantity that can be used as a measure in reference to the whole.</td>
<td rowspan="2">Spatial Visualisation
Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999).</td>
<td rowspan="2">Child 1
Child 2
Child 3
Child 4
Child 5</td>
<td>Child 6
Child 7
Child 8
Child 9</td>
</tr>
<tr><td>*Child 23
Child 25
(introduced later in the cycle for gesture observations – see Chapter 4)</td></tr>
<tr><td colspan="5">Relationship between fraction ideas and spatial constructs:
Visualising partitive division/ recursive multiplication between parts/ shares and whole. Conceiving the change in size of share as more shares are required.
Visualising shares involving mixed numbers.</td></tr>
<tr><td colspan="5">Task:
Introduce the picture book – “The doorbell rang” by Pat Hutchins. Ask the children to describe what is happening in the story.</td></tr>
</table>

Each child receives a "story board" that shows how many children were at the table at each part of the story. The children are asked to model/ draw how each group of cookies would be shared in each of the boxes.
Story board (A3 size):

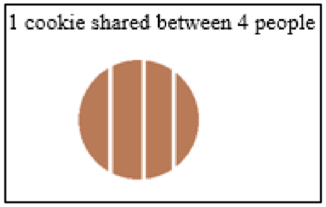
12 cookies, 2 children	12 cookies 4 children
12 cookies 6 children	16 cookies between 2 children

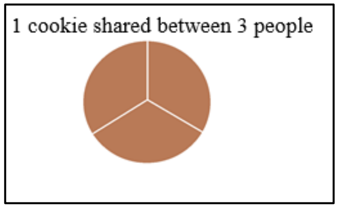
Children are provided with paper circles (as cookies) and plastic counters if they choose to use them.
Children are asked to name how they might describe the different shares of cookies.

Pilot Task 2: Creating Fair shares				
Rational Number Foci		**Spatial Reasoning Foci**	**Children trialling the task**	
Fraction as Measure	**Fraction as an operator**	**Construct**	**Group 1**	**Group 2**
Many-as-one Considering smaller units as a unit of one: ten ones as 1 ten, 2- quarters as 1 half etc) ***Unit Fraction*** Constructing and comparing unit fractions (1/*n*) though drawing on *many-as-one* idea (Confrey, 2008) ***Composite units*** Derived from splitting, a composite unit is a unit of units. E.g., 3/4 is a composite unit of three, 1/4 units as a result of a from a three split, 1 1/2 is a composite of two 3/4 units derived from halving; related to recursive multiplication.	***Fair shares*** The creation of equal size shares (of discrete collections or continuous wholes) where the shares created exhaust the whole. Naming fair shares of collections, including counting and relational naming (naming one share in relation to the whole collection or single whole). ***Doubling and Halving*** Derived from splitting (2-split) which is founded on repeated halving (splits of splits). Perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015). ***Partitive division/ recursive multiplication*** Partitive division: Connects to the process of fair sharing by starting with the dividend, then distributing one unit in each set and repeating this process until no more distributions can be shared fairly. ***Equipartitioning a whole*** Geometrical reasoning in which symmetries and congruence are utilised to develop equal parts of (primarily) rectangles and circles. ***Geometric Symmetries*** Geometric symmetries relate to fair share and equal parts/ groups. "When folding, congruence is built directly into the activity through symmetries, but the result of the action is hidden until the paper is unfolded, providing opportunities to examine one's predictions" (Confrey, 2012, p. 167) ***1-nth-of...*** The relationship between "1/nth of" in naming fair shares and identifying the referent units (greater or less than 1) for the fair shares resulting from equipartitioning (Confrey & Maloney, 2010, p. 973).	***Spatial Visualisation*** Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999). ***Mental Rotation*** "Mental rotation is the ability to imagine how an object would look in a different orientation – in other words, to turn something in one's mind" (Frick et al., 2013, p. 386).	Child 10 Child 11 Child 12 Child 13	Child 14 Child 15 Child 16 Child 17 Child 18 Child 21 Child 22 (introduced in part for gesture observations)

Relationship between fraction ideas and spatial constructs:
To partition small sets and continuous models to develop awareness of the size of the parts and number of parts created
To build visual recognition and awareness of the structure, form, pattern, and regularity of many-as-one parts forming a unit measure.
To visualise the act of partitioning to create other partitioning (splits) to conceptualise *1/nth of …*
The exploration of the different fraction ideas lies in asking children to consider the quantities from different perspectives (Wilson et al., 2012).

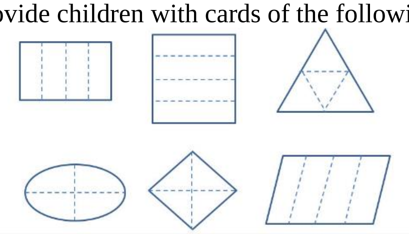
Task 2A:
Does each person get a fair share of cookies, in each of the following examples? How do you know?





Ask students to use counters as a scaffold.
Task 2B:
How many different ways can you share 12 cookies fairly? Can you add any new ways to what we did yesterday? Or 16? Or your own chosen number of cookies?

Task 2C:
Provide children with cards of the following images:



What is different about each shape, and what is the same? (Focus on proportional relationships)
What do these shapes and their parts have to do with fractions? Are there other ways these shapes can be shared fairly? Children may choose to draw representations to describe their thinking
Refer back to the non-example above and ask the children to explain what is the same and different, how the parts relate to fractions.

Pilot Task 3: Visualising the Share of a Cookie				
Rational Number Foci		**Spatial Reasoning Foci**	**Children trialling the task**	
Fraction as Operator	**Fraction as Measure**	**Construct**	**Group 1**	**Group 2**
Doubling Halving Derived from splitting (2-split) which is founded on repeated halving (splits of splits). Perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015). ***Partitive division/ Recursive multiplication*** Partitive division: Connects to the process of fair sharing by starting with the dividend, then distributing one unit in each set and repeating this process until no more distributions can be shared fairly. For example: 20 stickers shared between 4 children (Confrey & Scarano, 1995). For young children, partitive division often involves trial and error in creating fair shares, rather than problems that involve multiplicative reasoning (Hackenberg & Tillema, 2009). Recursive multiplication: The reversal of equipartitioning is reassembly or recursive multiplication (not counting; Confrey, 2012). For example: the introduction to division and multiplication are viewed as inverse operations to establish a recursive rather than iterative foundation for multiplication (i.e., times as many; Confrey et al., 2014b). ***Equi-partitioning a single whole***	***Many-as-one*** Many-as-one is a group of *m* objects, where the quotient represents the extensive quantity that one sharer receives (Confrey, 2012) For example: *If I share 12 lollies with my friend, we each get 6 lollies, 6 is half of the 12.* ***Composite units*** Derived from splitting, a composite unit is a unit of units. E.g., $\frac{3}{4}$ is a composite unit of three, $\frac{1}{4}$ units as a result of a from a three split, $1\frac{1}{2}$ is a composite of two $\frac{3}{4}$ units derived from halving; related to recursive multiplication.	***Spatial Visualisation*** Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations *within* objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999). The intent of this spatial construct for this study is to develop children's visualisation capabilities in relation to partitioning, unitising and equivalence concepts in a range of discrete and continuous contexts. For example, children will be encouraged to visualise the size and shape (*geometric symmetries* and *similarities*) and arrangements (*composite units, part-whole, many-to-one* or *many-as-one*) of a *fair share* by mentally manipulating and transforming objects or sets of objects. In addition, the process of creating *a fair share* (e.g., visualising *partitive division/ recursive multiplication, doubling/ halvin*g) of one quantity can be visualised and compared to another related quantity (*fraction equivalence, part-whole*) or unrelated quantities (*measure*) as examples. ***Spatial Proportional Reasoning*** Non-symbolic, visual recognition that shape, object and arrangements of different wholes can have the same value and therefore are equivalent. This can be an awareness of *doubling and halving, times as many, distribution* and *proto-ratio* ideas in the development of fraction magnitude. Spatial proportional reasoning includes *scaling*, which refers to the ability to compare different-sized spaces (Frick & Möhring, 2016); the ability to relate distances in one space to distances in another	Child 19 Child 20 Child 21 Child 22 Child 23	Child 24 Child 25 Child 26 Child 1 Child 2
				Child 15 Child 13 (introduced in part for gesture observations)

Geometrical reasoning in which symmetries and congruence are utilised to develop equal parts of (primarily) rectangles and circles. ***Geometric symmetries*** Geometric symmetries relate to fair share and equal parts/ groups. “When folding, congruence is built directly into the activity through symmetries, but the result of the action is hidden until the paper is unfolded, providing opportunities to examine one’s predictions” (Confrey, 2012, p. 167) ***Similarity*** Related to splitting through identifying similarities between the properties of equal shares and non-symbolic proportional relationships (continuous parts and sets)		space (Frick & Newcombe, 2012). *Spatial scaling* and *proportional scaling* recruit overlapping cognitive processes (Möhring et al., 2018) therefore *spatial proportional reasoning* for the purposes of this thesis includes the ideas of both spatial scaling and non-symbolic proportional reasoning. At this age, it is the perceptual awareness of this relationship, rather than necessarily quantitative measures. For example, transforming one space in size to match the other (Frick & Newcombe, 2012) such as “mentally shrink[ing] or expand[ing] spatial information in the sense of zooming in or out (of the map)…internally transforming magnitude information” (Möhring et al., 2018, p. 58). Thus, *geometric symmetries* and *similarity* of spaces, objects and arrangements are key connections in the development of fraction understanding.		

Relationship between fraction ideas and spatial constructs:
To visualise the act of partitioning and the multiplicative nature of doubling the parts results in halving the size of each share.
To estimate and visualise proportional similarities between different geometric wholes and their fractional parts.

Task:
Imagine what a cookie would look like if we had to share between two, then four then eight people? Can you imagine and predict what might happen to that cookie as each group of visitors arrives? *What if there were 3 people to share between? Then 6? What would those cookies look like if shred fairly? Just by looking at your cookie, can you visualise a way you would share your cookie fairly?*

Pilot Task 4: Sharing Easily Divisible Collections

Rational Number Foci			**Spatial Reasoning Foci**	**Children trialling the task**	
Fraction as a relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**	**Group 1**	**Group 2**
Many-to-One Many-to-one correspondence is the understanding of part-part relations (n:1). (Confrey & Smith, 1995). "Many" = counterpart objects, "One" = target object. For example: Three flowers for each vase (Sophian & Madrid, 2003). To make a juice mixture, there is a relationship between the water and juice quantities, which may not be equal, but preserved when replicated. ***Distribution*** Coordinating units to represent consistent part-part relations, with multiple target objects For example: Three children each receive two apples; In continuous contexts, recognising which part represents more than/ less than half. E.g., a container with juice and water may have more/ less water than juice; the water maybe more/ less than half of the container capacity. Naming the relationship such as 3 quarters water, 1 quarter juice produces a weaker mix than 3 quarters juice to 1 quarter water.	***Doubling and Halving*** Derived from splitting (2-split) which is founded on repeated doubling and halving (splits of splits), perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015). ***Partitive division/ recursive multiplication*** The introduction to division and multiplication are viewed as inverse operations ***1-nth-of...…*** The relationship between "1/nth of" in naming fair shares and identifying the referent units (greater or less than 1) for the fair shares resulting from equipartitioning (Confrey & Maloney, 2010, p. 973). ***Times as many*** Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using "times as many," or "times as much," (Confrey & Maloney, 2015, p. 924).	***Many-as-one*** Many-as-one is a group of *m* objects, where the quotient represents the extensive quantity that one sharer receives (Confrey, 2012) ***Unit fractions*** Unit fraction involves identifying and naming a single share of n fair shares as "1/n of 1" (Confrey & Maloney, 2015). ***Composite units*** Derived from splitting, a composite unit is a unit of units. E.g., 3/4 is a composite unit of 3 1/4 units - related to recursive multiplication.	***Spatial Visualisation*** Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999).	Child 3 Child 4 Child 5 Child 6	Child 7 Child 8 Child 9 Child 10 Child 11 Child 16 Child 18 (introduced in part for gesture observations)

Relationship between fraction ideas and spatial constructs:
Visualising partitive division/ recursive multiplication between parts/ shares and whole. Conceiving the change in size of share as more shares are required.
Naming 2 thirds as *two shares of…, two times as many-as-one share etc.*

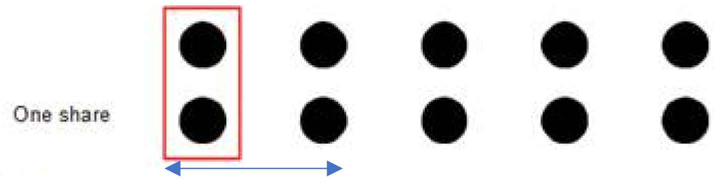
Task:

We know that the number of shares we partition our whole into, names the fraction. Using any number of counters under 12, divide your collections and see if you can record what fraction you have divided your set into. Record how you have done this, and name each of the shares (Model recording strategies as a whole group as many have difficulty with discrete sets).

Emphasise visualising the relationship/ action of partitive division and recursive multiplication

Prompts for children: when sharing 10, by 5 people… i.e., fifths; the whole is how many times as big as one share? How many cookies is three shares? (Confrey & Hotchkiss – Scarano, 1995).

What if I doubled the collection I started with? What changes? What stays the same?



Two shares/ 2-fifths:

Three shares/ 3-fifths / three *times as many-as-one* share.

Pilot Task 5: Cookie fraction estimation

<table>
<tr><th colspan="2">Rational Number Foci</th><th>Spatial Reasoning Foci</th><th colspan="2">Children trialling the task</th></tr>
<tr><th>Fraction as an operator</th><th>Fraction as Measure</th><th>Construct</th><th>Group 1</th><th>Group 2</th></tr>
<tr>
<td>Doubling and Halving
Derived from splitting (2-split) which is founded on repeated doubling and halving (splits of splits), perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015).</td>
<td>Unit fractions
Unit fraction involves identifying and naming a single share of n fair shares as “1/n of 1” (Confrey & Maloney, 2015).

Composite units
Derived from splitting, a composite unit is a unit of units. E.g., 3/4 is a composite unit of 3 1/4 units - related to recursive multiplication.

Part-whole fractions
The conceptualisation of the relationship between measure, many-as-one, composite and unit fractions for 1. For example: An apple is cut into y equal parts and x of these parts are eaten (Tsay & Hauk, 2009).</td>
<td>Mental Rotation
“Mental rotation is the ability to imagine how an object would look in a different orientation – in other words, to turn something in one’s mind” (Frick et al., 2013, p. 386).</td>
<td>Child 12
Child 13
Child 14
Child 15
Child 16</td>
<td>Child 17
Child 18
Child 19
Child 20
Child 21</td>
</tr>
</table>

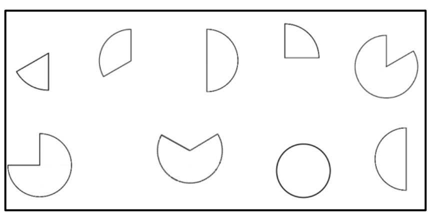
Relationship between fraction ideas and spatial constructs:

To visualise the quantities created from composite fractions through splitting (doubling/ halving, *n* splits)

Comparing images of cookie parts via mental rotation

Task:

Victoria and Sam left the cookie jar open one afternoon, and a mouse got in! It ate some of the cookies – here are what is left:



*Which cookie had been eaten the **most**?*

*Can you order the left-over cookies from the **least** eaten to the **most** eaten?*

Can you see any parts that would fit together to make a whole cookie? Name and describe the parts you join to make a whole cookie.

<table>
<tr><th colspan="6">Pilot Task 6: Birthday Cake Decorations</th></tr>
<tr><th colspan="2">Rational Number Foci</th><th>Spatial Reasoning Foci</th><th colspan="2">Children trialling the task</th></tr>
<tr><th>Fraction as an operator</th><th>Fraction as Measure</th><th>Construct</th><th>Group 1</th><th>Group 2</th></tr>
<tr><td>Doubling and Halving
Derived from splitting (2-split) which is founded on repeated doubling and halving (splits of splits), perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015).

Times as many
Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using “times as many,” or “times as much,” (Confrey & Maloney, 2015, p. 924).</td><td>Unit fractions
Unit fraction involves identifying and naming a single share of n fair shares as “1/n of 1” (Confrey & Maloney, 2015).

Composite units
Derived from splitting, a composite unit is a unit of units. E.g., 3/4 is a composite unit of 3 1/4 units - related to recursive multiplication.

Part-whole fractions
The conceptualisation of the relationship between measure, many-as-one, composite and unit fractions for 1.

Equivalent fractions
The equivalence of two fractional parts.
For example: $\frac{2}{4} = \frac{1}{2}$ (Confrey & Maloney, 2015), explored non-symbolically in this context</td><td>Mental Rotation
“Mental rotation is the ability to imagine how an object would look in a different orientation – in other words, to turn something in one’s mind” (Frick et al., 2013, p. 386).</td><td>Child 22
Child 23
Child 24
Child 25
Child 26</td><td>Child 1
Child 2
Child 3
Child 4
Child 5</td></tr>
<tr><td colspan="5">Relationship between fraction ideas and spatial constructs:
To visualise the quantities created from composite fractions through splitting (doubling/ halving, n splits)
Comparing images of cookie parts via mental rotation
Recognising times-as-many and double and halving structures to create different quantities of cookie (including mixed fractions (i.e., 3 halves as three times as many)</td></tr>
</table>

Task:

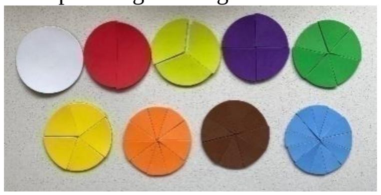
Victoria and Sam’s friend were having a birthday. They made her a birthday cake and cut up some of Ma’s cookies to decorate it on top.
So not to waste any of the leftover cookies, Ma put the unused parts back in the cookie tin. How many whole cookies are there? (Adapted from Mix, Levine, and Huttenlocher (1999).

To scaffold: teacher models putting cookie fractions into a tin – e.g., 3 halves, 1 half at a time. Once each fractional part is put in the tin the students won’t be able to see them anymore. 2- halves (one after another) – describe how much of a cookie?

- 2 quarters
- 3 quarters
- 1 half and 1 quarter
- 3 halves

Students visualise what they think the parts would look like all put together and describe how much. To scaffold, children might like to nominate *more than*, *less than* a whole, half etc.

Each partner gets a bag with fractional parts:

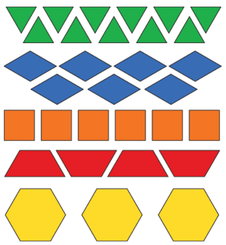


The must show the parts to their partner one at a time, as they put in a cookie jar (i.e., paper bag). Their partner must mentally keep track of parts and see if they can describe how many cookies those parts would make. Again, the child can draw how many whole cookies would be made if all the parts were placed together and nominate *more than*, *less than* a whole, half etc.

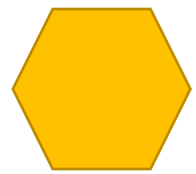
Pilot Task 7: Patten Block Fractions					
Rational Number Foci			**Spatial Reasoning Foci**	**Children trialling the task**	
Fraction as a relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**	**Group 1**	**Group 2**
Proto-ratio Coordinating two numerical sets, typically through *building up* and *building down* strategies (Hino & Kato 2019). Building up: *If there are 6 lollies in one packet, how many lollies in 3 packets?* Building down: *If there are 18 lollies in three packets, how many lollies are sold in one packet?* Continuous example: 4 equal parts water, 2 equal parts juice Similarly, responses such as: *"3 out of 12, or "1 out of every 4"* are examples of the proto-ratio idea (Confrey & Maloney, 2015, p. 925).	***Double Halving*** Derived from splitting (2-split) which is founded on repeated doubling and halving (splits of splits), perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015). ***Geometric symmetries*** Geometric symmetries relate to fair share and equal parts/ groups. "When folding, congruence is built directly into the activity through symmetries, but the result of the action is hidden until the paper is unfolded, providing opportunities to examine one's predictions" (Confrey, 2012, p. 167) ***Times as many*** Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using "times as many," or "times as much," (Confrey & Maloney, 2015, p. 924). ***Similarity*** Related to splitting through identifying similarities between the properties of equal shares and proportional relationships (continuous parts and sets)	***Unit fractions*** Unit fraction involves identifying and naming a single share of n fair shares as "1/n of 1" (Confrey & Maloney, 2015). ***Composite units*** Derived from splitting, a composite unit is a unit of units. E.g., 3/4 is a composite unit of 3 1/4 units - related to recursive multiplication. ***Part-whole fractions*** The conceptualisation of the relationship between measure, many-as-one, composite and unit fractions for 1. ***Equivalent fractions*** The equivalence of two fractional parts For example: $\frac{2}{4} = \frac{1}{2}$ (Confrey & Maloney, 2015). Explored non-symbolically via perception of geometric models (size/ proportions of parts) and discrete sets.	***Spatial Visualisation*** Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999). ***Spatial Proportional Reasoning*** Non-symbolic, visual recognition that shape, object and arrangements of different wholes can have the same value and therefore are equivalent.	Child 6 Child 7 Child 8 Child 9 Child 10	Child 11 Child 12 Child 13 Child 14 Child 15
Relationship between fraction ideas and spatial constructs: Children will use pattern blocks to manipulate and create fractional parts.					

A focus on operating on children’s pattern block construction and establishing the relational proportions between the representations. E.g., What would it look like if it were only half the size? Double the size? Three times etc.

Task:
The children will have access to a range of pattern blocks, e.g.:



Using the pattern blocks, students need to model 1 half, 1 quarter, 1 third and a free choice fraction.



For example: *If this is a whole:*
What is half?
1-third?
Three times as much?

Children are to make and record their findings. Children can then experiment making their own creations and demonstrating fractional parts/ scaling/ ratios etc.
Describe how you constructed your representations. What fractions did you create? What strategies did you use to help you? How did you shrink or enlarge your original picture? What patterns did you discover?

<table>
<tr><th colspan="5">Pilot Task 8: Who Ate More Pizza?</th></tr>
<tr><th colspan="2">Rational Number Foci</th><th>Spatial Reasoning Foci</th><th colspan="2">Children trialling the task</th></tr>
<tr><th>Fraction as an operator</th><th>Fraction as Measure</th><th>Construct</th><th>Group 1</th><th>Group 2</th></tr>
<tr>
<td>Equipartitioning a single whole
Geometrical reasoning in which symmetries and congruence are utilised to develop equal parts of (primarily) rectangles and circles.

Similarity
Related to splitting through identifying similarities between the properties of equal shares and non-symbolic proportional relationships (continuous parts and sets)

Times as many
Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using “times as many,” or “times as much,” (Confrey & Maloney, 2015, p. 924).
For example: How many times as large as one share is the whole collection? (reassembly)…the whole collection is 3 times as much as one share” (Confrey & Maloney, 2015, p. 922).</td>
<td>Measure
Directly related to fair share, in that when fair shares are created, these shares represent a quantity that can be used as a measure in comparison to the whole.

Composite Units
Derived from splitting, a composite unit is a unit of units. E.g., 3/4 is a composite unit of three, 1/4 units as a result of a from a three split, 1 1/2 is a composite of two 3/4 units derived from halving; related to recursive multiplication.

Equivalent Fractions
The equivalence of two fractional parts
For example: 2/4 = 1/2 (Confrey & Maloney, 2015).
Explored non-symbolically via geometric models and discrete sets in this study.</td>
<td>Spatial Visualisation
Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999).

Spatial Proportional Reasoning
Non-symbolic, visual recognition that shape, object and arrangements of different wholes can have the same value and therefore are equivalent.</td>
<td>Child 16
Child 17
Child 18
Child 19
Child 20</td>
<td>Child 21
Child 22
Child 23
Child 24
Child 26
Child 1</td>
</tr>
<tr><td colspan="5">Relationship between fraction ideas and spatial constructs:

To visualise the act of partitioning and relationship between splitting to conceptualise fraction measures
Children will visually recognise and compare the size of the parts in relation to the whole and compare between each child.</td></tr>
<tr><td colspan="5">Task:
Ma ordered 2 pizzas for tea. Both pizzas are the same size but when Victoria and Sam opened the boxes, one was cut into fourths, and one was cut into eighths.
If you were really hungry and wanted the biggest slice of pizza, which pizza would you take a slice from – 1- fourth or 1-eighth? Why? (Children can use mini whiteboards and markers to problem solve).
Sam claimed he ate more than Victoria because he took two slices from the pizza partitioned into eighths. Victoria ate only one slice from the pizza that was cut into fourths. Who ate more? How do you know?

What are some other ways you can cut your pizzas, so Sam and Victoria eat a different number of slices, but eat the same amount? (2 fourths and 1 half for example).

Stretch: Victoria and Sam share 1 pizza. Victoria has eaten twice as much pizza as Sam. What size parts could they have eaten?</td></tr>
</table>

<table>
<tr><th colspan="7">Pilot Task 9: Tablecloths</th></tr>
<tr><th colspan="3">Rational Number Foci</th><th>Spatial Reasoning Foci</th><th colspan="2">Children trialling the task</th></tr>
<tr><th>Fraction as an operator</th><th></th><th>Fraction as Measure</th><th>Construct</th><th>Group 1</th><th>Group 2</th></tr>
<tr>
<td>Distribution
Coordinating units to represent consistent part-part relations, with multiple target objects
For example: Three children each receive two apples; In continuous contexts, recognising which part represents more than/ less than half. E.g., a container with juice and water may have more/ less water than juice; the water maybe more/ less than half of the container capacity. Naming the relationship such as 3 quarters water, 1 quarter juice produces a weaker mix than 3 quarters juice to 1 quarter water.</td>
<td>Double Halving
Derived from splitting (2-split) which is founded on repeated doubling and halving (splits of splits), perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015).

Partitive division/ Recursive Multiplication
Partitive division: Connects to the process of fair sharing by starting with the dividend, then distributing one unit in each set and repeating this process until no more distributions can be shared fairly.
For example: 20 stickers shared between 4 children (Confrey & Scarano, 1995). For young children, partitive division often involves trial and error in creating fair shares, rather than problems that involve multiplicative reasoning (Hackenberg & Tillema, 2009).

Recursive multiplication: The reversal of equipartitioning is reassembly or recursive multiplication (not counting; Confrey, 2012).
For example: the introduction to division and multiplication are viewed as inverse operations to establish a recursive rather than iterative foundation for multiplication (i.e., times as many; Confrey et al., 2014b). Involves splitting and reassembly of continuous models and discrete sets.

Geometric symmetries
Geometric symmetries relate to fair share and equal parts/ groups. “When folding,</td>
<td>Composite units
Derived from splitting, a composite unit is a unit of units. E.g., 3/4 is a composite unit of 3 1/4 units - related to recursive multiplication.

Part-whole fractions
The conceptualisation of the relationship between measure, many-as-one, composite and unit fractions for 1.

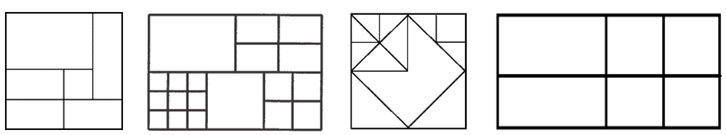
Equivalent fractions
The equivalence of two fractional parts
For example: $\frac{2}{4} = \frac{1}{2}$ (Confrey & Maloney, 2015). Explored non-symbolically via perception of geometric models (size/ proportions of parts) and discrete sets.</td>
<td>Spatial Visualisation
Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999).

Mental Rotation
“Mental rotation is the ability to imagine how an object would look in a different orientation – in other words, to turn something in one’s mind” (Frick et al., 2013, p. 386).

Spatial Proportional Reasoning
Non-symbolic, visual recognition that shape, object and arrangements of different wholes can have the same value and therefore are equivalent.</td>
<td>Child 25
Child 2
Child 3
Child 4
Child 5</td>
<td>Child 6
Child 7
Child 8
Child 9
Child 10</td>
</tr>
</table>

<table>
<tr><td></td><td>congruence is built directly into the activity through symmetries, but the result of the action is hidden until the paper is unfolded, providing opportunities to examine one's predictions" (Confrey, 2012, p. 167)

Similarity
Related to splitting through identifying similarities between the properties of equal shares and proportional relationships (continuous parts and sets)</td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="6">Relationship between fraction ideas and spatial constructs:
Children will visually recognise and compare (through mental rotation) the size of parts and proportion of colour within and between tablecloths
Exploring the process of multiple mental folding and or splitting (SV) and rotating parts of the tablecloth to determine proportions of colour, comparing regions of incongruent and congruent wholes. Noticing distributions of composite units and comparing part-part and part-whole relationships.</td></tr>
<tr><td colspan="6">Task:
Ma wanted to buy a new tablecloth for the kitchen table. She asked Victoria and Sam to go to the shops and see if they could find one that was suitable. She asked for it to be in the colours of purple and orange, but she wanted it to be more purple than orange. Victoria and Sam found the following tablecloths. Which of the tablecloths can Sam and Victoria choose from? How much is the purple part in each cloth? How do you know?

Here are some new tablecloths Ma was considering buying. Choose one and colour it to represent half purple and half blue, or 1 quarter yellow etc. How many other equal parts can you see? Can you name the parts?
</td></tr>
</table>

Pilot Task 10: Mapping					
Rational Number Foci			**Spatial Reasoning Foci**	**Children trialling the task**	
Fraction as a relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**	**Group 1**	**Group 2**
Distribution Coordinating units to represent consistent part-part relations, with multiple target objects For example: *Three children each receive two apples;* In continuous contexts, recognising which part represents more than/ less than half. E.g., a container with juice and water may have more/ less water than juice; the water maybe more/ less than half of the container capacity. Naming the relationship such as 3 quarters water, 1 quarter juice produces a weaker mix than 3 quarters juice to 1 quarter water.	***Double Halving*** Derived from splitting (2-split) which is founded on repeated doubling and halving (splits of splits), perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015). ***1-nth-of….*** The relationship between "1/nth of" in naming fair shares and identifying the referent units (greater or less than 1) for the fair shares resulting from equipartitioning (Confrey & Maloney, 2010, p. 973). ***Scaling*** Related to *times as many*. "There is only one salient dimension here, namely, objects. The splitting operation in this instance establishes the foundation of the ideas of a scalar (a dimensionless number possessing only magnitude) and a scaling factor" (Confrey, 2012, p. 162). ***Times as many*** Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using "times as	***Unit fractions*** Unit fraction involves identifying and naming a single share of n fair shares as "1/n of 1" (Confrey & Maloney, 2015). ***Composite units*** Derived from splitting, a composite unit is a unit of units. E.g., 3/4 is a composite unit of 3 1/4 units - related to recursive multiplication. ***Part-whole fractions*** The conceptualisation of the relationship between measure, many-as-one, composite and unit fractions for 1.	***Spatial Visualisation*** Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999). ***Spatial Proportional Reasoning (Scaling)*** Non-symbolic, visual recognition that shape, object and arrangements can have the same value and therefore are equivalent. ***Mental rotation*** Mental Rotation is the ability to imagine how an object would look if it were rotated; that is, mentally turn a 2D or 3D object (Frick et al., 2013).	Child 11 Child 12 Child 13 Child 14	Child 15 Child 16 Child 17 Child 18 Child 19

	many," or "times as much," (Confrey & Maloney, 2015, p. 924).				

Relationship between fraction ideas and spatial constructs:

Visualising different pathways on the carpet maps by observing the roads and paths an imagining how far a dinosaur has travelled through these spaces.

Scaling distances from carpet mat to pictorial representations

Visualising and comparing the spatial structure and size of different parts (linear paths/ areas) between the physical carpet models and pictorial representations.

Creating scaled representations of fractional paths, naming and describing the distributions of the measures: half of …path is greater than half of another path.

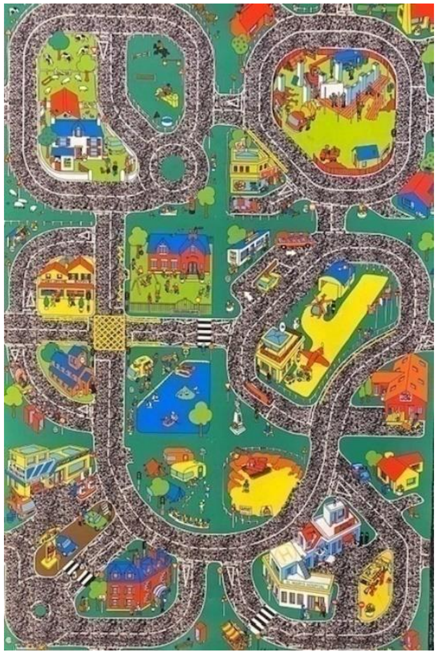
Task:

Introduce the new picture book "Knock, knock dinosaur" by Caryl Hart.

The dinosaurs have escaped the boys house! They've decided to explore the neighbourhood – here is the map. Somebody said they saw a T-Rex halfway between the boy's house, and the zoo. Where would that be? (The Food store). Another person said they saw a dinosaur halfway between the central fountain and the duck pond – where would that be? (museum). One lady saw a velociraptor 2- thirds of the way along the road in front of the café, heading toward the food market…where would this dinosaur be?

Lots of scaffolding of directions/ position and then visualising 2/3 of the length of said road.

Can you represent a road and landmarks on your individual whiteboards (spatial scaling foci).



You have taken a helicopter out to see if you can find some dinosaurs. On this map of the town (floor carpet), match the dinosaurs to their locations by sticking the correct tag on each dinosaur and placing on the mat.
Children represent parts of the carpet map that described where they saw each dinosaur, on an A3 sheet of paper. They need to draw the points of interest – e.g., the runway of the airport; draw the position of the dinosaur and then write in words their explanation such as, "dinosaur spotted *1 quarter* of the way along the runway, twice the length of the fence towards the hospital".

Children can draw a map of another's whole (or part thereof) mat, again describing the position of the dinosaur in a fraction sense.
What is the same about the carpet… (zoo/ airport/ farm etc) and your map? (Same proportions/ fraction, different scale etc). What is different? (Absolute size).

Pilot Task 11: Hidden Fractions

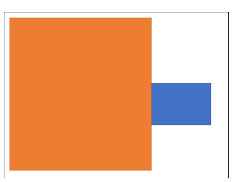
Rational Number Foci		Spatial Reasoning Foci	Children trialling the task	
Fraction as an operator	**Fraction as Measure**	**Construct**	**Group 1**	**Group 2**
Double Halving Derived from splitting (2-split) which is founded on repeated doubling and halving (splits of splits), perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015). ***1-nth-of……*** The relationship between "1/nth of" in naming fair shares and identifying the referent units (greater or less than 1) for the fair shares resulting from equipartitioning (Confrey & Maloney, 2010, p. 973). ***Times as many*** Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using "times as many," or "times as much," (Confrey & Maloney, 2015, p. 924).	***Many-as-one*** Many-as-one is a group of *m* objects, where the quotient represents the extensive quantity that one sharer receives (Confrey, 2012) ***Composite units*** Derived from splitting, a composite unit is a unit of units. E.g., 3/4 is a composite unit of 3 1/4 units - related to recursive multiplication. ***Unit fractions*** Unit fraction involves identifying and naming a single share of n fair shares as "1/n of 1" (Confrey & Maloney, 2015).	***Spatial Visualisation*** Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999).	Child 14 Child 16 Child 17 Child 18	Child 15 Child 19 Child 20 Child 21 Child 22

Relationship between fraction ideas and spatial constructs:

Children make conjectures about the proportion and size of the blue part that could be hidden. This includes reasoning if more than/ less than half is hidden or showing and drawing on their spatial visualisation skills to determine what splits could be made.

Task:

What fraction of the blue rectangle could be hidden underneath the orange square? How many possibilities are there? Visualise, try out a strategy, and justify.

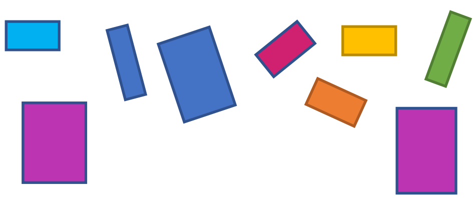


Pilot Task 12: Chocolate Ratios

Rational Number Foci		Spatial Reasoning Foci	Children trialling the task	
Fraction as a relation	**Fraction as a measure**	**Construct**	**Group 1**	**Group 2**
Many-to-one correspondence is the understanding of part-part relations (*n*:1). (Confrey & Smith, 1995). "Many" = counterpart objects, "One" = target object. For example: *Three flowers for each vase* (Sophian & Madrid, 2003) ***Distribution*** Coordinating units to represent consistent part-part relations, with multiple target objects For example: *Three children each receive two apples*	***Many-as-one*** Many-as-one is a group of *m* objects, where the quotient represents the extensive quantity that one sharer receives (Confrey, 2012) For example: *If I share 12 lollies with my friend, we each get 6 lollies, six is half of the 12.*	***Spatial Visualisation*** Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999).	Child 20 Child 21 Child 22 Child 23 Child 24	Child 25 Child 26 Child 1 Child 2 Child 3

Relationship between fraction ideas and spatial constructs:
Exploring the connection between many-as-one and many-to-one ideas by naming and renaming the collection. *Many-to-one* requires to the child to name how many per person/ per share, with distribution ideas between part (children) and part (chocolates) explored, whilst *many-as-one* considers a share of the chocolates as a fraction of the whole set, 1 share is a third of the set. It involves is visualising the compositions of the quantities involved to name and rename the shares.

Task: *I have some big blocks of chocolate and some small blocks. If I were to share the following chocolate bars between three people so everyone gets some big and small blocks how might I do that, so it is fair? What is a rule I could invent for sharing these chocolates (each share is a ratio of 1 big block: 2 small bars)? Describe each person's share in relation to the whole set.*



Pilot Task 13: The French Fry Task				
Rational Number Foci		**Spatial Reasoning Foci**	**Children trialling the task**	
Fraction as an operator	**Fraction as Measure**	**Construct**	**Group 1**	**Group 2**
Partitive Division/ Recursive Multiplication Partitive division: Connects to the process of fair sharing by starting with the dividend, then distributing one unit in each set and repeating this process until no more distributions can be shared fairly. For example: 20 stickers shared between 4 children (Confrey & Scarano, 1995). For young children, partitive division often involves trial and error in creating fair shares, rather than problems that involve multiplicative reasoning (Hackenberg & Tillema, 2009). Recursive multiplication: The reversal of equipartitioning is reassembly or recursive multiplication (not counting; Confrey, 2012). For example: the introduction to division and multiplication are viewed as inverse operations to establish a recursive rather than iterative foundation for multiplication (i.e., times as many; Confrey et al., 2014b). Involves splitting and reassembly of continuous models and discrete sets. ***Equipartitioning a whole*** Geometrical reasoning in which symmetries and congruence are utilised to develop equal parts of (primarily) rectangles and circles.	***Unit fractions*** Unit fraction involves identifying and naming a single share of *n* fair shares as "1/n of 1" (Confrey & Maloney, 2015).	***Spatial Visualisation*** Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999).	Child 23 Child 24 Child 25 Child 26 Child 1	Child 2 Child 3 Child 4 Child 5 Child 6
Relationship between fraction ideas and spatial constructs: This tasks was originally designed by Tzur (2019) from a measurement/ iterative approach to teaching fractions, however I re-focused the task to explore it from a splitting approach that emphasised visualising the act of splitting and reassembly in relation to the whole (Partitive division/ recursive multiplication) as well as an awareness of fair share, congruence through the geometric properties of the "French fry".				
Tzur's "French fry" modified task.				

Mum bought home Maccas for the dinosaurs for tea one night – lucky dinosaurs! But when she got home, they had only put in a small pack of fries to share between everyone!

Children will be given different lengths of yellow tape to represent a French fry.

Task 13A: *Can you share this fry equally between 2 dinosaurs?*
Attach to the child's partitioning operations (observable through folding—child may fold initially.
Tell me about your strategy. Why did you fold the paper into two parts? What is the name of each part you created? How can you convince me that they are halves?

Task 13B*: Share one fry equally among three people. Indicator: Promote the child's splitting operations through spatial visualisation of parts.*
Questions: Have you achieved thirds? Why/ why not? What do you notice about thirds here in relation to halves in your previous task?

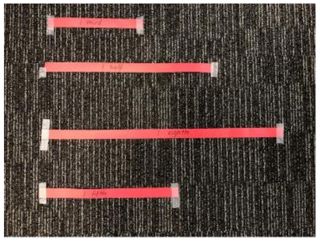
Task 13C: *Share one fry equally among five people*
Within task questions*: I see you created unequal shares. How can you ensure fair shares? Before you make a guess about the size of the share among five people, look at the size of the shares when we shared among three people. Will sharing between five result in bigger or smaller shares? Describe how you know*

Pilot Task 14: Finding Wholes

Rational Number Foci		**Spatial Reasoning Foci**	**Children trialling the task**	
Fraction as an operator	**Fraction as a measure**	**Construct**	**Group 1**	**Group 2**
1-nth-of…… The relationship between "1/nth of" in naming fair shares and identifying the referent units (greater or less than 1) for the fair shares resulting from equipartitioning (Confrey & Maloney, 2010, p. 973). ***Times as many*** Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using "times as many," or "times as much," (Confrey & Maloney, 2015, p. 924).	***Composite Units*** Derived from splitting, a composite unit is a unit of units. E.g., $\frac{3}{4}$ is a composite unit of three, $\frac{1}{4}$ units as a result of a from a three split, 1 $\frac{1}{2}$ is a composite of two $\frac{3}{4}$ units derived from halving; related to recursive multiplication. ***Part-whole*** The conceptualisation of the relationship between *measure, many-as-one, composite* and *unit fractions* for 1.	***Spatial Visualisation*** Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999).	Child 7 Child 8 Child 9 Child 10 Child 11	Child 12 Child 13 Child 14 Child 15 Child 16

Relationship between fraction ideas and spatial constructs:
Estimating and visualising the whole (length) based on a fractional part. To explore the times as many, part-whole and composite unit relationship.

Task:
On the floor, place different lengths of masking tape with a size on them. *If this is half, a fourth, a third etc, what is one whole?* Estimate and visualise first where the total length of the whole would be. Then, come up with a strategy that will help you. What does the ***name*** of the part tell us about how many parts are in the whole? How close were your estimations of the whole to the actual length? What made it easy to visualize? What made it hard?

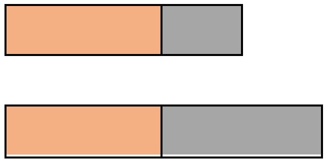


Pilot Task 15: Muesli Bars

Rational Number Foci		Spatial Reasoning Foci	Children trialling the task	
Fraction as an operator	**Fraction as Measure**	**Construct**	**Group 1**	**Group 2**
Doubling / Halving Derived from splitting (2-split) which is founded on repeated halving (splits of splits). Perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015). ***Times as Many*** Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using "times as many," or "times as much," (Confrey & Maloney, 2015, p. 924). For example: How many times as large as one share is the whole collection? (reassembly)…the whole collection is 3 times as much as one share" (Confrey & Maloney, 2015, p. 922). ***1-nth-of...*** The relationship between "1/nth of" in naming fair shares and identifying the referent units (greater or less than 1) for the fair shares resulting from equipartitioning (Confrey & Maloney, 2010, p. 973). For example: *12 objects shared among 3 children, a share is 4 objects (Per child). Each child receives* $\frac{1}{3}$ *of the collection; one third of 12 is 4* (Confrey & Maloney, 2015).	***Composite units*** Derived from splitting, a composite unit is a unit of units. E.g., $\frac{3}{4}$ is a composite unit of three, $\frac{1}{4}$ units as a result of a from a three split, $1\frac{1}{2}$ is a composite of two $\frac{3}{4}$ units derived from halving; related to recursive multiplication. ***Unit fractions*** Unit fraction involves identifying and naming a single share of *n* fair shares as "1/n of 1" (Confrey & Maloney, 2015).	***Spatial proportional reasoning*** Spatial proportional reasoning is the non-symbolic, visual recognition that shape, object and arrangements of different wholes can have the same value and therefore are *equivalent.*	Child 17 Child 18 Child 19 Child 20 Child 21	Child 22 Child 23 Child 24 Child 25 Child 26

Relationship between fraction ideas and spatial constructs:
Visualising and preserving continuous part-part quantities. Estimating fractional parts of the bars that have been eaten.

Task:
Mum keeps a few boxes of muesli/ chocolate bars in the pantry for snacks. They are not all the same size. From each pair of muesli bars, which bar did the T-Rex eat the MOST of? How do you know? Which bar did it eat the LEAST of? How do you know?
(Children will receive 2 bars in various sizes with various amount eaten – as indicated by a different colour e.g., grey section uneaten, brown section eaten etc). They need to work out which bar out of the two had *more* eaten.



Student can try and verbalise how much of each bar was eaten.

What did you discover? How did you decide which bar had more of it eaten? What strategies did you use to work this out?

Pilot Task 16: How Many Steps?				
Rational Number Foci		**Spatial Reasoning Foci**	**Children trialling the task**	
Fraction as a measure	**Fraction as an operator**	**Construct**	**Group 1**	**Group 2**
Many-to-one Many-to-one correspondence is the understanding of part-part relations (*n*:1). (Confrey & Smith, 1995). "Many" = counterpart objects, "One" = target object. For example: *Three flowers for each vase* (Sophian & Madrid, 2003). To make a juice mixture, there is a relationship between the water and juice quantities, which may not be equal, but preserved when replicated. ***Distribution*** Coordinating units to represent consistent part-part relations, with multiple target objects For example: *Three children each receive two apples;* In continuous contexts, recognising which part represents more than/ less than half. E.g., a container with juice and water may have more/ less water than juice; the water maybe more/ less than half of the container capacity. Naming the relationship such as 3 quarters water, 1 quarter juice produces a weaker mix than 3 quarters juice to 1 quarter water. ***Proto-ratio*** Coordinating two numerical sets additively, typically through *building up* and *building down* strategies (Hino & Kato 2019). Building up: *If there are 6 lollies in one packet, how many lollies in 3 packets?* Building down: *If there are 18 lollies in three packets, how many lollies are sold in one packet?* Continuous example: 4 equal parts water, 2 equal parts juice Similarly, responses such as: *"3 out of 12, or "1 out of every 4"* are examples of the proto-ratio idea (Confrey & Maloney, 2015, p. 925).	***Partitive division/ recursive multiplication*** Partitive division: Connects to the process of fair sharing by starting with the dividend, then distributing one unit in each set and repeating this process until no more distributions can be shared fairly. For example: *20 stickers shared between 4 children* (Confrey & Scarano, 1995). For young children, *partitive division* often involves trial and error in creating fair shares, rather than problems that involve multiplicative reasoning (Hackenberg & Tillema, 2009). Recursive multiplication: The reversal of equipartitioning is reassembly or recursive multiplication (not counting; Confrey, 2012). For example: the introduction to division and multiplication are viewed as inverse operations to establish a recursive rather than iterative foundation for multiplication (i.e., *times as many*; Confrey et al., 2014b). Involves splitting and reassembly of continuous models and discrete sets. ***Times as many*** Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using "times as many," or "times as much," (Confrey & Maloney, 2015, p. 924). For example: *How many times as large as one share is the whole collection? (reassembly)…the whole collection is 3 times as much as one share"* (Confrey & Maloney, 2015, p. 922).	***Spatial Visualisation*** Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999).	Child 1 Child 2 Child 3 Child 4 Child 5	Child 7 Child 8 Child 9 Child 10 Child 11
Relationship between fraction ideas and spatial constructs: Children explore discrete part-part relations of footsteps, i.e., dinosaur: human quantities. Focus is on visualising the complexity of the many-to-one relationship in the outset (i.e., *for each* Dino step, we take many steps to travel the <u>same</u> distance). Naming the relationship as *times-as-many* and building up/ building down to represent pro-ratios.				

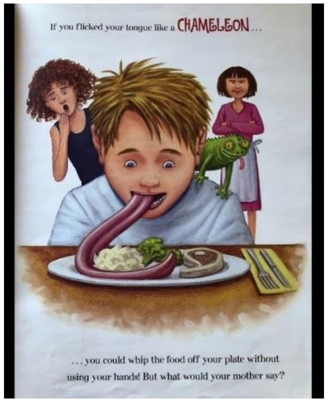
Task: *If one dinosaur step was equal to two of your normal steps, how many of your steps would you need to take for five dinosaur steps?* *A dinosaur only took 1 quarter of the number of steps it took you to walk across the park. How many steps did you each take? How many possibilities are there?* *What if the dinosaur had taken six steps, and you had taken eighteen? What is the smallest number of steps you would need to take for one dinosaur step? How can we represent this in a way that helps us describe what is happening?* Children will have access to concrete materials such as blocks and counters and paper to record pictorially if they wish. *What did you discover? How did you problem solve this question and represent your answer?* *If you took 5 normal steps and a dinosaur took 5 normal steps, who would go further? How much further? Why?*

Pilot Task 17: Animal Proportions				
Rational Number Foci		**Spatial Reasoning Foci**	**Children trialling the task**	
Fraction as Operator	**Fraction as Measure**	**Construct**	**Group 1**	**Group 2**
Doubling / Halving Derived from splitting (2-split) which is founded on repeated halving (splits of splits). Perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015). ***Times as Many*** Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using "times as many," or "times as much," (Confrey & Maloney, 2015, p. 924). For example: How many times as large as one share is the whole collection? (reassembly)…the whole collection is 3 times as much as one share" (Confrey & Maloney, 2015, p. 922). ***1-nth-of...*** The relationship between "1/nth of" in naming fair shares and identifying the referent units (greater or less than 1) for the fair shares resulting from equipartitioning (Confrey & Maloney, 2010, p. 973). For example: *12 objects shared among 3 children, a share is 4 objects (per child). Each child receives* $\frac{1}{3}$ *of the collection; one third of 12 is 4* (Confrey & Maloney, 2015).	***Composite units*** Derived from splitting, a composite unit is a unit of units. E.g., $\frac{3}{4}$ is a composite unit of three, $\frac{1}{4}$ units as a result of a from a three split, $1\frac{1}{2}$ is a composite of two $\frac{3}{4}$ units derived from halving; related to recursive multiplication. ***Unit fractions*** Unit fraction involves identifying and naming a single share of *n* fair shares as "1/n of 1" (Confrey & Maloney, 2015).	***Spatial proportional reasoning*** Spatial proportional reasoning is the non-symbolic, visual recognition that shape, object and arrangements of different wholes can have the same value and therefore are *equivalent.*	Child 6 Child 12 Child 13 Child 14 Child 15	Child 16 Child 18 Child 19 Child 20 Child 21

Relationship between fraction ideas and spatial constructs:
Exploring and preserving continuous part-part quantities and relationship when replicated (enlarged/ shrunk).

Task:

Read a selection of pages from picture book, *If you hopped like a frag by* David M. Schwartz (1999*)*
Some animals have some pretty interesting characteristics. For example, a Chameleon's tongue is twice the length of their bodies. (Sample page from picture book):



If *you were a chameleon – but the same height you are now, how long would your tongue be? Discuss strategies you would use to figure this out* (i.e., model using freeze tape to measure heights and partition in half).

Using strips of paper, make your tongue based on the chameleon's proportions (twice the length of your body = length of tongue). *What if the length of the Chameleon's tongue was only one third of the length of its body? How many times is your body the length of your tongue?*

If *you were a frog, you could hop five times the length of your leg. How far could you hop if you were a frog? Estimate first, think about what strategies you would use.*

Provide students with different length of freeze tape: *If this was a frog jump, how long would its leg be? How do you know? (*Visualising fractions of a measure).
What do the names of the fraction parts tell us about the size of the part?

Pilot Task 18: Plant Growth Rate

Rational Number Foci		Spatial Reasoning Foci	Children trialling the task	
Fraction as a relation	**Fraction as Operator**	**Construct**	**Group 1**	**Group 2**
Distribution Coordinating units to represent consistent part-part relations, with multiple target objects For example: *Three children each receive two apples;* In continuous contexts, recognising which part represents more than/ less than half. E.g., a container with juice and water may have more/ less water than juice; the water maybe more/ less than half of the container capacity. Naming the relationship such as 3 quarters water, 1 quarter juice produces a weaker mix than 3 quarters juice to 1 quarter water	***Doubling / Halving*** Derived from splitting (2-split) which is founded on repeated halving (splits of splits). Perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015).	***Spatial proportional reasoning*** The ability to compare different-sized spaces or measures (Frick & Möhring, 2016); the ability to relate distances in one space to distances in another space (Frick & Newcombe, 2012). In this case it is visualising a consistent whole in which to imagine and compare the rate of growth against. E.g., imagining which plant is bigger at *either* the half year or full year mark.	Child 17 Child 22 Child 23 Child 24 Child 25	Child 26 Child 1 Child 2 Child 3 Child 4

Relationship between fraction ideas and spatial constructs:
To recognise the need for a consistent measure to compare the growth rate (half a year or one year) and then visualise the growth of each the plant in relation to that measure.

Task:

Plant A grew 5cm tall in *half a year*.

Plant B grew 8cm tall in a *whole year.*

Which plant is growing faster? How do you know?

Images are provided as distractors, so children need to focus on visualising a consistent measure to compare each plant.

<table>
<tr><th colspan="5">Pilot Task 19: Dino Paths</th></tr>
<tr><th colspan="2">Rational Number Foci</th><th>Spatial Reasoning Foci</th><th colspan="2">Children trialling the task</th></tr>
<tr><th>Fraction as Operator</th><th>Fraction as Measure</th><th>Construct</th><th>Group 1</th><th>Group 2</th></tr>
<tr><td>Equi-partitioning a single whole
Geometrical reasoning in which symmetries and congruence are utilised to develop equal parts of (primarily) rectangles and circles.

Scaling
Related to times as many. “There is only one salient
dimension here, namely, objects. The splitting operation in this instance establishes the foundation of the ideas of a scalar (a dimensionless number possessing only magnitude) and a scaling factor” (Confrey, 2012, p. 162).</td><td>Unit Fractions
Unit fraction involves identifying and naming a single share of n fair shares as “1/n of 1” (Confrey & Maloney, 2015).</td><td>Spatial proportional reasoning (Scaling)
The ability to compare different-sized spaces (Frick & Möhring, 2016); the ability to relate distances in one space to distances in another space (Frick & Newcombe, 2012). Spatial scaling and proportional scaling recruit overlapping cognitive processes (Möhring et al., 2018) therefore spatial proportional reasoning for the purposes of this thesis includes the ideas of both spatial scaling and non-symbolic proportional reasoning.</td><td>Child 17
Child 22
Child 23
Child 24
Child 25</td><td>Child 26
Child 1
Child 2
Child 3
Child 4</td></tr>
<tr><td colspan="5">Relationship between fraction ideas and spatial constructs:
Estimating the fraction of the oval each of the dinosaurs travelled, and then comparing that in proportion to the oval. Using spatial proportional reasoning (possibly benchmarking to half) to determine which paths is proportionally longer.</td></tr>
<tr><td colspan="5">Task:
You took a helicopter out to look for the dinosaurs who had escaped and took photos of different areas where the dinosaurs had been seen.

Here is a photo of the school oval that was taken from different heights in the helicopter. What do you notice about the different photos of the ovals? Which was taken as a close up? Further away?</td></tr>
</table>

The lines on the ovals indicate how far two different dinosaurs walked across the ovals. Which dinosaur walked the furthest? How do you know?

Pilot Task 20: Bags of Wool				
Rational Number Foci		**Spatial Reasoning Foci**	**Children trialling the task**	
Fraction as an operator	**Fraction as Measure**	**Construct**	**Group 1**	**Group 2**
Fair share The creation of equal size shares (of discrete collections or continuous wholes) where the shares created exhaust the whole. Naming fair shares of collections, including counting and relational naming (naming one share in relation to the whole collection or single whole). For example: *12 objects shared among 3 children, a share is 4 objects* *(per child). Relationally, each child receives $\frac{1}{3}$ of the col-* *Lection* (Confrey & Maloney, 2015, p. 924)	***Measure*** Directly related to fair share, in that when fair shares are created, these shares represent a quantity that can be used as a measure in comparison to the whole. ***Composite Units*** Derived from splitting, a composite unit is a unit of units. E.g., 3/4 is a composite unit of three, 1/4 units as a result of a from a three split, 1 1/2 is a composite of two 3/4 units derived from halving; related to recursive multiplication.	***Spatial Visualisation*** Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999).	Child 17 Child 22 Child 23 Child 24 Child 25	Child 26 Child 1 Child 2 Child 3 Child 4
Relationship between fraction ideas and spatial constructs: Visualising the outcome of sharing five bags fairly between three people, and the size of each share. Encouraging naming of the share (visualising whether each person would get more than/ less than …)				
Task: Recite the nursery rhyme, Ba Ba Black Sheep. *If the sheep produced three bags of wool – one for the master, one for the dame, and one for the little boy, how much wool each person would receive if they had to share three bags between five people? Can you estimate/ visualise approximately what each share will be?* (E.g., at least one full bag and some more…)				

<table>
<tr><th colspan="5">Pilot Task 21: Cuisenaire Fractions</th></tr>
<tr><th colspan="2">Rational Number Foci</th><th>Spatial Reasoning Foci</th><th colspan="2">Children trialling the task</th></tr>
<tr><th>Fraction as an operator</th><th>Fraction as a measure</th><th>Construct</th><th>Group 1</th><th>Group 2</th></tr>
<tr>
<td>Times as Many
Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using “times as many,” or “times as much,” (Confrey & Maloney, 2015, p. 924). For example: How many times as large as one share is the whole collection? (reassembly)…the whole collection is 3 times as much as one share” (Confrey & Maloney, 2015, p. 922).

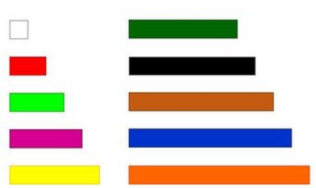
Similarity
Related to splitting through identifying similarities between the properties of equal shares and non-symbolic proportional relationships (continuous parts and sets)</td>
<td>Composite units
Derived from splitting, a composite unit is a unit of units. E.g., $\frac{3}{4}$ is a composite unit of three, $\frac{1}{4}$ units as a result of a from a three split, $1\frac{1}{2}$ is a composite of two $\frac{3}{4}$ units derived from halving; related to recursive multiplication.

Unit fractions
Unit fraction involves identifying and naming a single share of n fair shares as “1/n of 1” (Confrey & Maloney, 2015).</td>
<td>Spatial proportional reasoning
Spatial proportional reasoning is the non-symbolic, visual recognition that shape, object and arrangements of different wholes can have the same value and therefore are equivalent.
This can be an awareness of doubling and halving, times as many, distribution, and proto-ratio ideas in the development of fraction magnitude

Spatial Visualisation
Spatial visualisation is the ability or skill drawn upon to imagine multi-step spatial transformations within objects or sets of objects (Frick, 2019; Lowrie et al., 2021; Linn & Petersen, 1985; Sorby, 1999).</td>
<td>Child 5
Child 6
Child 7
Child 8
Child 10</td>
<td>Child 9
Child 11
Child 12
Child 13
Child 14</td>
</tr>
<tr><td colspan="5">Relationship between fraction ideas and spatial constructs:
Visualising the relationship between various measures / rods and naming their fractional relationships. Visualize the splitting action and the relationship between the number of shares and their size.</td></tr>
</table>

Task:

Children will have sets of Cuisenaire rods to explore.



If this is one (orange, dark green etc) – what colour is half? (Spatial proportion activity)
If this is a 2 thirds (dark green) what is one whole?
What is 3 times light green? How many ways can you name it?

<table>
<tr><th colspan="5">Pilot Task 22: Dinosaur Versus Human</th></tr>
<tr><th colspan="2">Rational Number Foci</th><th>Spatial Reasoning Foci</th><th colspan="2">Children trialling the task</th></tr>
<tr><th>Fraction as an operator</th><th>Fraction as a measure</th><th>Construct</th><th>Group 1</th><th>Group 2</th></tr>
<tr>
<td>Doubling / Halving
Derived from splitting (2-split) which is founded on repeated halving (splits of splits). Perceptually recognising similarity (Confrey & Smith, 1995). Conceptualising the size of the share(s) as twice as big for doubling, or two times as small/ half as big for halving (Confrey & Maloney, 2015).

Times as Many
Related to reassembly (recursive multiplication). The ability to name the original collection multiplicatively in relation to a single fair share using “times as many,” or “times as much,” (Confrey & Maloney, 2015, p. 924).
For example: How many times as large as one share is the whole collection? (reassembly)…the whole collection is 3 times as much as one share” (Confrey & Maloney, 2015, p. 922).</td>
<td>Composite units
Derived from splitting, a composite unit is a unit of units. E.g., $\frac{3}{4}$ is a composite unit of three, $\frac{1}{4}$ units as a result of a from a three split, 1 $\frac{1}{2}$ is a composite of two $\frac{3}{4}$ units derived from halving; related to recursive multiplication.

Unit fractions
Unit fraction involves identifying and naming a single share of n fair shares as “1/n of 1” (Confrey & Maloney, 2015).</td>
<td>Spatial proportional reasoning
Spatial proportional reasoning is the non-symbolic, visual recognition that shape, object and arrangements of different wholes can have the same value and therefore are equivalent.
This can be an awareness of doubling and halving, times as many, distribution and proto-ratio ideas in the development of fraction magnitude</td>
<td>Child 15
Child 16
Child 17
Child 21
Child 23</td>
<td>Child 24
Child 18
Child 25
Child 26
Child 1</td>
</tr>
<tr><td colspan="5">Relationship between fraction ideas and spatial constructs:
Visualising the relationship between various measures / rods and naming their fractional relationships. Visualize the splitting action and the relationship between the number of shares and their size.</td></tr>
<tr><td colspan="5">Task:
All of the dinosaurs are different shapes and sizes. To illustrate, we can make a type of reference that will help us explain how big each dinosaur is, compared to a human.

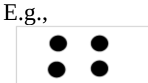
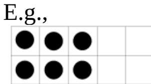
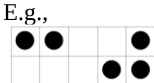
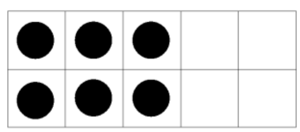
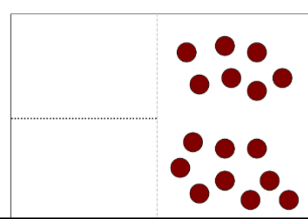
Draw each of the dinosaurs and yourself to represent the following proportions:

• A diplodocus is six times taller than an average adult (like your teacher’s height).
• A velociraptor is half the height of an average adult.
• An adult is 1 quarter of the height of a t-rex.</td></tr>
</table>

- *An adult is half as tall as a triceratops.*

What strategies did you use to draw your pictures? What is the same or different between yours and your friends pictures?

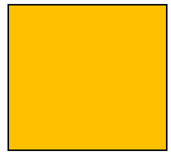
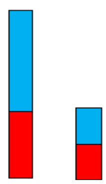
Appendix B: Task-Based Interview Items

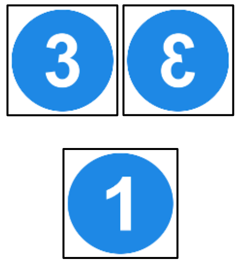
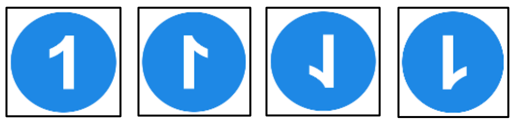
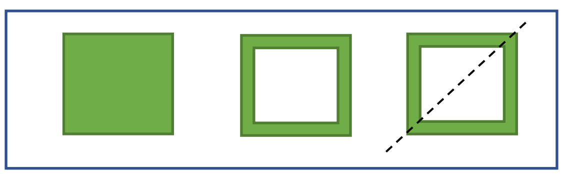
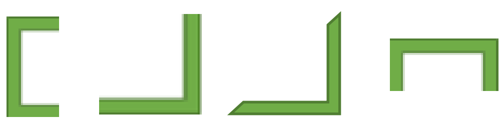
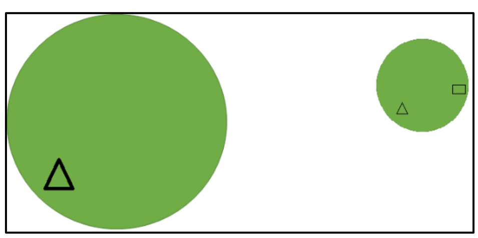
Set One: Trusting the Count			
Item	**Conceptual Focus**	**Materials/ Stimulus For each Task**	**Source and Description**
1: Subitising collections	*Part-part-whole knowledge*	Cards 1-6 (common dot die arrangement) E.g.,  Cards 7-10 (ordered arrangements) E.g.,  Cards 7- 19 (tens frames ordered and random) E.g., 	Hold the Subitising Cards out of the view of the child. Show each card in order for approximately 2 seconds. After each card, ask, *"How many dots were on that card? … How did you work that out"* (Siemon, 2006)
2: Hidden counters task	*Part-part-whole knowledge*	9 Counters Calico bag (Example of resources provided)  	Place 5 counters and bag in front of child, rattle to demonstrate that there are counters in the bag. Place four counters in front of child. *"There are 4 counters here and 5 more in this bag. How many counters altogether? How did you work that out?"* (Siemon, 2006)
3: Tens frame Bananas	*Part-part-whole knowledge*	Tens frame representing 6 bananas: 	Children are asked to think about the dots on the tens frames as bananas. *"If I have this many bananas, and three more bananas were added, how many are there altogether?"* (Siemon, 2006)
4: Hidden Dots	*Part-part-whole knowledge*	Dot card with flap: (Cut along dotted line to fold bottom flap over) 	Place the '7' and '9' dot cards in front of the child. *"There are 7 dots here and 9 dots here.* Turn the '9' card over and ask: *How many dots altogether? How did you work that out?"* (Siemon, 2006)
Set Two: Place Value			
5: 26 counters	*Place value parts Composite, countable units*	26 counters plastic counters (Example image of resource provided) 	Tip out counters and say, *"Please count these and write down how many".* Circle the 6 in 26 and ask, *"Does this have anything to do with how many counters you have there?"* Circle the 2 in 26 and repeat the question. Ask child to explain their thinking if not obvious. (Siemon, 2006)

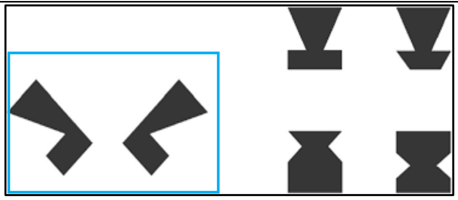
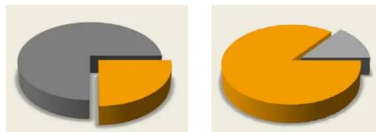
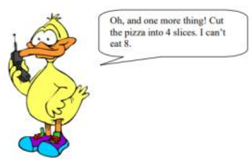
6: Place-value Bundles	*Place value parts* *Composite, countable units*	13 bundles of ten pop sticks and 16 single sticks (Example image of physical resources provided) 	Place bundles and single sticks in front of the student, point to bundles and say, *"Do you know how many straws are in this bundle?*" Suggest counting if necessary, then say, *"Please show me how you would make 34"*. If child asks or moves to unbundle a ten, point to the bundles of ten and say, *"Before you do that, is there any way you could use these to make 34?"* (Siemon, 2006)
7: More than/ Less than.. task	*Place value parts* *Composite, countable units*	More than/ Less than Number Card: 86	Place the card in front of the student and ask, *"Write the number that is 1 more than this number? Write the number that is 1 ten more than this number?"* If correct, say, *"Write the number that is 3 less than this number? Write the number that is 2 tens more than this number?*" Ask child to explain their thinking if not obvious. (, Siemon, 2006).
8: Proportional Number line task	*Place value ordering* *Spatial proportional reasoning*	Number line card. Card is folded on the dotted line so only one number line is visible at once. 0________________________________20 (fold line) - 0______________________________100___ ___________________________	Place the 0 to 20 half of the folded Open Number Line Card in front of the child and say, "*Use the pencil to make a mark to show where you think the number 8 would be …Why did you put it there?"* Repeat with the number 16. If reasonably accurate and/or explanation plausible, turn the card over to show the 0 to 100 open number line and say, *"Make a mark to show where you think 48 would be. Why did you put it there?"* Repeat with the numbers 67 and 26. Ask child to explain their thinking if not obvious. (Siemon, 2006)
9: Four-digit number task	*Place value parts* *Composite, countable units*	Four-digit number card: 5308	Child is presented with card "*Can you please read this number?"* If correct, say *"Can you count on by ones please? What is ten less?* (Siemon, 2006)

Set Three: Fractions and Spatial Reasoning

10: Folding Fractions	*Fair share* *Doubling/ halving* *Unit fractions*	Image of a square:	Child is shown the image of a square. *"How many ways can you imagine folding a square in half? Can you*

	Spatial visualisation		*describe what you think it would look like if you folded it in half, then in half again? What is each part called?"* University of Cambridge, 1997-2023)
11: What fraction is green?	*Composite units* *Unit fractions* *Many-to-one (2 fifths: 3 fifths)* *Spatial proportional reasoning*	Image of rectangle: 	Child is shown the rectangle and asked, "*What fraction of this rectangle is shaded green? How did you work that out?"* (Created by researcher)
12: Comparing unit fractions	*Unit fractions* *Equipartitioning a single whole* *1-nth-of……* *Spatial Visualisation*	No additional stimulus provided.	Researcher asks: *"Which is bigger, 1 third or 1 eighth? How do you know?"* (Created by researcher)
13: Cordial mixture	*Scaling* *Distribution* *Spatial proportional reasoning*	Image of two "jugs "of water and cordial mixtures: 	Child is shown the two jugs of cordial mixture and asked, *"Which jug of cordial will taste sweeter? How do you know?"* Adapted from Boyer & Levine, 2012)
14: Bags of wool	*Many-to-one* *Many-as-one* *Partitive division/ recursive multiplication* *Spatial visualisation*	No image. Referred children to the well-known nursery rhyme, *Ba Ba Black Sheep.*	Child is shown the pictorial stimulus and asked, *"Each sheep produces 3 bags of wool What if there were five sheep, and they each produced three bags of wool? How many bags?* (Siemon et al., 2011)
15: Missing faces	*Composite Units* *1-nth-of.....* *Spatial visualisation*	Set image: 	Each child is shown the set and asked, *"If this is 2-thirds of a set of stickers, what is missing? Draw or explain your thinking"* (Created by researcher)
16: Halving the stars	*Doubling and halving* *1-nth-of……* *Geometric symmetries* *Many-as-one* *Spatial visualisation*	Problem image: 	Child is presented with the image. "*If you gave away half of this collection of stars, how many would you have left?*" (Created by researcher)

17: Flip it	*Geometric symmetries* *Mental rotation*	Problem image:  Choose one of the following options: 	The child is shown the image of the pairs of “3s”. *“If we flipped this number sideways, it would look like this”.* The child is asked, *“What would the number 1 look like (point to number 1) if we flipped it sideways?”* (Adapted from Nrich.maths.org)
18: Giselle’s paper square	*Doubling and halving* *Geometric symmetries* *Equipartitioning a single whole* *Spatial Visualisation*	Images provided for children whilst researcher is explaining the task:  	A series of folds is made to a square, and the child needs to identify what the end result would be from four possible options. *“Gisele had a green sheet of paper and cut a white shape out of the middle of the paper. Then she folds the paper in half, diagonally. Which of the four shapes below did Gisele see?”* Note: Images of the paper square being cut and folded not shown to child. (Adapted from Ekstrom et al., 1976)
19: Scale the picture	*Doubling and Halving* *Times as many* *Geometric symmetries* *Scaling* *Spatial proportional reasoning*	Spatial scaling image: 	An image of two circles is presented. The first circle has the diameter doubles, and area increased by four times. Children are required to draw the missing element (rectangle) and describe a relationship between the two images. *“Can you complete the picture of the circle on the left, so it has the same objects as the right-hand image? Explain why you chose to draw the objects in that way? Can you explain how these two circles are the same or different?* (Adapted from Frick & Möhring, 2016)
20: What’s the object?	*Doubling and halving* *Geometric symmetries*	Mental rotation image:	Children are required to choose an image that would be the outcome of combining 2 halves of an object. *“If you pushed the two parts in the blue*

	Equipartitioning a whole *Unit fractions* *Mental rotation*		*box together, which of the four shapes on the right would it make? Why do you think so?"* (Mix, 1998).
21: Eating pies	*Equipartitioning a whole* *Recursive multiplication* *Composite unit* *Unit fractions* *Part-whole fractions* *Spatial visualisation*	Images of pies: 	Children are presented with a 3D image of two "pies" that have a fractional part missing. They need to use their visualisation skills to determine what fraction each part represents. *"Can you describe what size each piece of pie is? How did you work that out?"* (Adapted from Way, 2011).
22: Fred's pizza	*Equipartitioning a whole* *Equivalent fractions* *Spatial visualisation*	Image of Fred ordering pizza. No image of an actual pizza supplied; however, children have access to pencils and paper for use. 	*"Fred orders a pizza which he will eat the whole thing in one sitting. But…he asks for it to be cut into quarters, not eighths, because he can't eat eight slices of pizza. Does his request make sense? Why or why not?"* (Adapted from Dole, 1999)
23: Plant growth rate	*Distribution* *Proto-ratio* *Doubling and halving* *Spatial Proportional Reasoning*	Image provided of the plants as a distractor:  Plant A  Plant B	Comparing rate of growth. *"If plant A grows 5cm in half a year, and plant B grows 8cm in a whole year, which is growing faster? How do you know"* (Adapted from Dole et al., 2012)
24: How big is a half?	*Measure* *Unit fractions* *1-nth-of……* *Spatial Visualisation*	No additional stimulus provided.	*"I want you to close your eyes and think about what you see when you think about the word half? Just think for a moment about when you might have heard that word. Now tell me, is half big or small? Tell me why you think that?"* (Adapted from Ball, 1993)

Appendix C: Intervention Program

Lesson 1: Sharing Cookies	**Rational Number Foci**			**Spatial Reasoning Foci**
	Fraction as Relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**
		Fair shares *Doubling/ Halving* *Partitive division* *1-nth-of……* *Geometric* *Symmetries*	*Many-as-one* *Measure;*	*Spatial Visualisation* **Added during Cycle Two: Spatial structuring of units/ shares created – i.e., pattern, formation, repetition of units.*
Relationship between fraction ideas and spatial constructs	Visualising partitive division/ recursive multiplication between parts/ shares and whole. Conceiving the change in size of share as more shares are required. Visualising shares involving mixed numbers. *Emphasising the structure and organisation of parts and wholes to help develop the mathematical ideas.			

Launch

Questions and provocations for the children: *What are fractions? What types of fractions have you heard of before? When you hear the word half, what do you think of? (Close your eyes and imagine). When you hear quarter? What do you see in your mind?*
Draw the pictures you see in your mind about half and quarter.
After children explore these questions and representations, they will share with each other in small groups.

Explore

Introduce the picture book – "The doorbell rang" by Pat Hutchins. Ask the children to describe what is happening in the story.
Each child receives a "story board" that shows how many children were at the table at each part of the story. The children are asked to model/ draw how each group of cookies would be shared in each of the boxes.
Story board (A3 size):

12 cookies, 2 children	12 cookies 4 children
12 cookies 6 children	8 cookies 12 children

Children are provided with paper circles (as cookies) and plastic counters if they choose to use them.
Children are asked to name how they might describe the different shares of cookies.

Summarise

Discuss how the children problem solved; specifically, 8 cookies between 12 children (a complex problem not explored in the picture book)
Intentional Teaching: *How much of the whole set of cookies (12) does 1 of these 8 children have? What patterns do you notice about the shares created?*
What happened to the number of cookies each person receives when there are more children to share the cookies with?
What did you notice about each person's share? What does this have to do with fractions?

Lesson 2: What is a Fair Share	**Rational Number Foci**			**Spatial Reasoning Foci**
	Fraction as Relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**
	Many-to-one	*Fair shares* *Doubling and Halving* *Partitive division/ recursive multiplication* *Geometric Symmetries* *1-nth-of...*	*Many-as-one* *Unit fractions* *Composite units*	*Spatial Visualisation* *Mental Rotation* **Added during Cycle Two: Spatial structuring*
Relationship between fraction ideas and spatial constructs	To partition small sets and continuous models to develop awareness of the size of the parts and number of parts created *To build visual recognition and awareness of the structure, form, pattern, and regularity of many-as-one parts forming a unit measure. To visualise the act of partitioning to create other partitioning (splits) to conceptualise fraction measures Exploring naming the collections from the fraction as relation, many-to-one perspective: e.g., x cookies *for each* child			

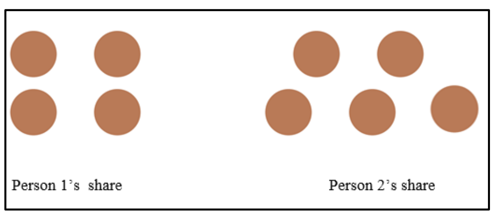
Launch

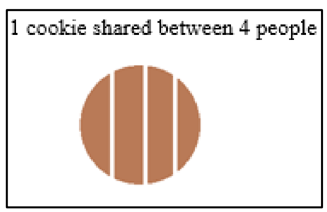
Recap initial ideas from Lesson 1. *What is a fraction? Where have you seen/ heard/ used fraction ideas in your daily life?*

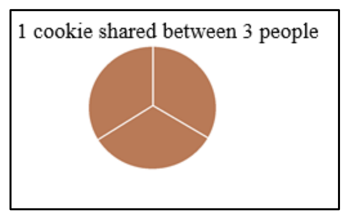
Each child takes a strip of paper (all different lengths). They are asked to partition in half . Then each child is asked to buddy up with a partner or group of three and compare.
Questions and Provocations:
What is the same? What is different? Have you all got half? Why or why not? What does half mean?

Explore

Re-read "The doorbell rang" by Pat Hutchins.
Small groups of children receive laminated sets of cards of each picture (they can draw with whiteboard markers on these).
Does each person get a fair share of cookies, in each of the following plates? (Start by modelling and discussing a continuous and discrete example as a whole class).
2 cookies shared between 2 people. Fair or unfair? Why?







Ask students to use counters as a scaffold.
How many different ways can you share 12 cookies fairly? Can you add any new ways to what we did yesterday? Or 16? Or your own chosen number of cookies?

What about sharing 1 cookie between a group of children? How could you do this fairly?

Summarise

Questions and Provocations:
Let's take a minute to go over a few important "rules" about fractions. When we talk about fraction names like halves ("twoths"), quarters (fourths), thirds, fifths… that means each of those parts or size of the groups are exactly the same.

If I were to cut this cookie like this (show non-example ) *can we call them fourths/ quarters? Why or why not? How do you know just by visualising?*

Children are asked to discuss their thinking with a set model of four unequal groups: three unequal groups with counters.

Invite children to cut up the cookie above to prove that they are not equal parts when cut this way.

So, let's talk about the name of some of the parts we explored today. What about this cookie:  *How many parts? Are they all equal? How do we know?* – e.g., engage in spatial visualisation/ mental rotation). *What do we call them? What does the number of equal parts tell us about the name of the fraction?*

How many cookies is a fair share if there are three people and 12 cookies? 16 cookies and 4 people?

What did you discover about your sharing examples? Was it easy to tell a fair share from an unfair share? Why?

<table>
<tr><td rowspan="3">Lesson 3: Visualising the share of a cookie</td><td colspan="3">Rational Number Foci</td><td>Spatial Reasoning Foci</td></tr>
<tr><td>Fraction as Relation</td><td>Fraction as Operator</td><td>Fraction as Measure</td><td>Construct</td></tr>
<tr><td></td><td>Doubling Halving
Partitive division/ Recursive multiplication
Equipartitioning a whole
Geometric symmetries
Similarity</td><td>Many-as-one
Composite units</td><td>Spatial Proportional Reasoning
Spatial Visualisation</td></tr>
<tr><td>Relationship between fraction ideas and spatial constructs</td><td colspan="4">Exploring proportional relationships between different shapes that have been partitioned into various fractions (i.e., quarters, halves, thirds). Reasoning how they represent the same fractions in relation to their whole.
To visualise the act of partitioning and the multiplicative nature of doubling the parts results in halving the size of each share.
To estimate and visualise proportional similarities between different geometric wholes and their fractional parts.</td></tr>
</table>

<u>Launch</u>

Take a square sheet of paper, fold into half (must be square or ‘short’ rectangle for this task for ease of folding diagonally).

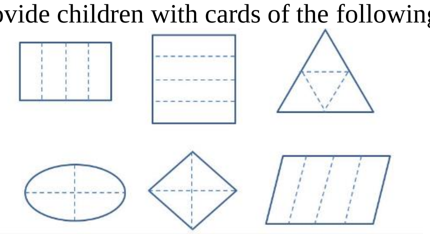
Questions and provocations:

If I wanted to fold it into fourths, what would that mean? How many ways can I do that so there are fourths (four equal parts)?

Model one example and also a non-example.

How many fourths are there in a whole (sheet of paper)? Why?

Provide children with cards of the following images:



What is different about each shape, and what is the same? (Focus on proportional relationships)

What do these shapes and their parts have to do with fractions?

Refer back to the non-example above and ask the children to explain what is the same and different, how the parts relate to fractions.

<u>Explore</u>

Bring out a large, paper cookie (A4 size).

If I were the only one in the room, how much of the cookie would be mine? (The whole thing).

What happened if someone else came along? (model halving the cookie – either by drawing on board or folding cookie). Focus on language and size of parts.

And the doorbell rang – its 2 more friends! Ask children to get into groups of four. *How will we equipartition to each get a fair share? What is each part called now? (fourth) Why?* (Name of the part names the share etc). *Imagine what your cookie would look like if you had to share between 8 people. Just by looking at your cookie, can you* ***visualise*** *a way you would share your cookie fairly? Now use your cookie to fold/ partition into eighths. Are all the parts equal?*

Finally, children were given a paper cookie/ lemon slice (rectangular shape) and asked to share between a number of their friends – three, six. They had to visualise and draw where and how they would cut their cookie.

Summarise

Describe what happens to the cookie as more people come to share the cookie. If you were hungry and it was your favourite cookie – would you like to share between 3 people or 5 people? That is, would you prefer a third of a cookie or a fifth? Visualise and discuss. What does this mean for fractions? Is there a rule we state that applies to fraction which helps us think about them? (The more shares the smaller the part).

Lesson 4: Sharing divisible collections	**Rational Number Foci**			**Spatial Reasoning Foci**
	Fraction as Relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**
	Many-to-one *Distribution*	*Partitive division/ recursive multiplication* *Times-as-many*	*Many-as-one* *Composite units* *Unit fractions*	*Spatial visualisation* **Added during Cycle Two: Spatial structuring*
Relationship between fraction ideas and spatial constructs	*Visualising the relationship/ action of partitive division and recursive multiplication between sets of objects. Naming and renaming the shares to build a connection between many-to-one, times-as-many and many-as-one shares.* *Spatial structuring of distributing far shares of collections.* *Use of spatial language (i.e., spatial dimensions).*			

Launch

Questions and Provocations:

Who can remember a rule for creating fractions? (The more shares the smaller the part/ share; all parts/ shares must be equal to be named a fraction / number of fair shares names the fraction).

Take a strip of paper, and fold in half. Open – what can you see? (2 <u>equal</u> parts). *What do we call them?* (Halves or twos)

Now fold in half again – what can you see? (Four *equal* parts). *What do we call them?* (Fourths) *What can you tell me about fourths and halves?* (Direct children's attention to one half = 2 equal parts, are bigger than 1 fourth/ quarter etc).

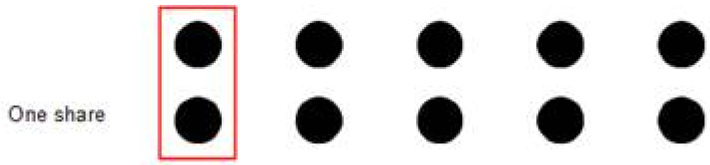
Imagine re-folding in half, half and then in half one more time. Predict what you think will happen – Check: What has happened? What can you tell me about the parts you see? What are they called? What can you tell me about 2- eighths?

Which is bigger? 1-eighth or 1- half? 1- fourth or 1- eighth?

What is the same here as what we did yesterday with the cookies? What is different? Emphasise that the shape of the whole to start with does not matter, it is the way its partitioned. That is, a circle and a square can both be partitioned into eighths, but they may look different. The fraction is relative to its whole – nothing else.

Explore

We know that the number of shares we partition our whole into, names the fraction. Using any number of counters under 12, divide your collections and see if you can name what fraction you have divided your set into. Record how you have done this, and name each of the shares (Model recording strategies as a whole group as many have difficulty with discrete sets). Emphasise visualising the relationship/ action of partitive division and recursive multiplication
Prompts for children: when sharing 10, by 5 people… i.e., fifths; the whole is how many times as big as one share? How many cookies is three shares? (Confrey & Hotchkiss – Scarano, 1995).



Two shares/ 2-fifths:

Three shares/3-fifths / three times as many-as-one share.

Summarise

Children will develop some conjectures or statements about the distribution of the set and how they have named the fractional parts**.** *What do you notice about the number of shares you have created, and how big each share is?*

Lesson 5: Cookie fraction estimation	**Rational Number Foci**			**Spatial Reasoning Foci**
	Fraction as Relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**
		Double halving *Times as many* *Similarity*	*Composite units* *Unit fractions* *Part-Whole fractions* *Equivalent fractions*	*Spatial Visualisation (Mental Rotation)*
Relationship between fraction ideas and spatial constructs	Visualising the magnitude of each part in relation to a whole, and the total quantity created (equal to and greater than one) through composite fractions. Comparing images of cookie parts via mental rotation. Recognising times-as-many and double and halving structures to create different quantities of cookie (including mixed fractions (i.e., 3 halves as three times as many)			

Launch
Victoria and Sam's friend were having a birthday. They made her a birthday cake and cut up some of Ma's cookies to decorate it on top.
So not to waste any of the leftover cookies, Ma put the unused parts back in the cookie tin.

How many whole cookies are there?

To scaffold: teacher models putting cookie fractions into a tin – e.g., 3 halves, one half at a time shown to the students in different orientations. Once each fractional part is put in the tin the students won't be able to see them anymore. Student can draw what they see one at a time, but the focus is on visualising the total quantity of the parts and connecting that to their understanding of composite parts.

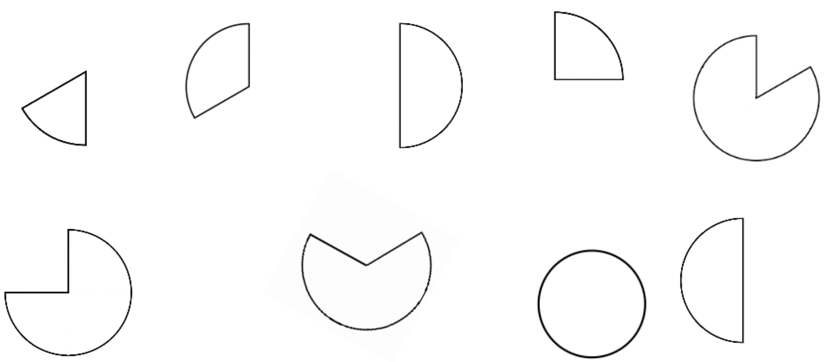
- 2 halves (one after another) – describe how much of a cookie?
- 2 quarters - how much?
- 3 halves
- 1 whole, 1 half and 1 quarter

Students visualise and can draw what they think the parts would look like all put together.

Each partner gets a bag with fractional parts but only whole, halves quarters and thirds are used for this task. One partner choose three parts. They must show the parts to their partner one at a time, as they put in a cookie jar (i.e., paper bag). Their partner must mentally keep track of parts or draw and see if they can describe the size of the parts. Then they need to work out the total quantity created form combining the three part, justify and then check with the parts.

Explore

Victoria and Sam left the cookie jar open one afternoon, and a mouse got in! It ate some of the cookies – here are what is left: (students will be shown each of these parts and asked to visualise and use the fraction kits to help them solve what fraction was eaten etc).



*Which cookie had been eaten the **most**?*

Can you see any parts that would fit together to make a whole cookie? Name and describe the parts you join to make a whole cookie. Children have access to fraction kits to explore this task

Summarise

What did you notice about the different parts? How do you know which pairs fit together? What strategies did you use? What parts were hard/ easy to visualise combining? What strategies did you use (estimating if it were more than/ less than half/ quarter etc?)

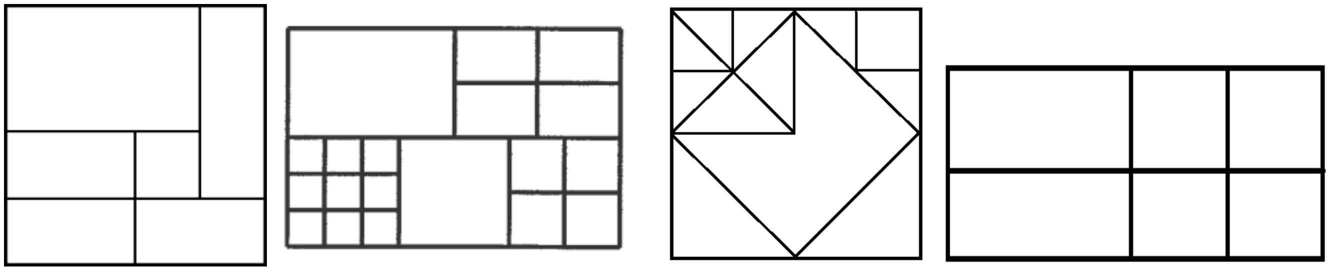
Lesson 6 Tablecloths	**Rational Number Foci**			**Spatial Reasoning Foci**
	Fraction as Relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**
	Distribution	*Doubling/ Halving* *Partitive Division/ Recursive multiplication* *Equi-partitioning a whole* *Geometric symmetries* *Similarity*	*Composite units* *Part-whole fractions* *Equivalent fractions*	*Spatial Visualisation* *Spatial Proportional Reasoning*
Relationship between fraction ideas and spatial constructs	Exploring the process of multiple mental folding (SV) and rotating parts of the tablecloth to determine proportions of colour, comparing regions of incongruent and congruent wholes. Noticing distributions of composite units and comparing part-part and part-whole relationships.			

Launch

Ma wanted to buy a new tablecloth for the kitchen table. She asked Victoria and Sam to go to the shops and see if they could find one that was suitable. She asked for it to be in the colours of purple and orange, but she wanted it to be ***more purple*** than orange. Victoria and Sam found the following tablecloths. *Which of the tablecloths can Sam and Victoria choose from? How much is the purple part in each cloth? How do you know?*

Explore

Here are some new tablecloths Ma was considering buying. Where can you see a half? A fourth? How many other fractions can you see? Colour and name the parts. (A range of simple and complex patterns will be provided).
Children will also be provided with blank squares and rectangles to create their own representations.



<u>Summarise</u>

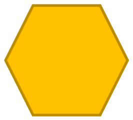
What fraction parts did you see? What strategies did you use to help you see different fraction parts?(multiple mental fold/ rotating parts, comparing proportional regions of incongruent and congruent wholes) What did you discover about the proportion of colour on your tablecloth?

Lesson 7 Pattern Block Fractions	**Rational Number Foci**			**Spatial Reasoning Foci**
	Fraction as Relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**
	Proto-ratio	*Doubling/ Halving* *Geometric symmetries* *Similarity* *Times as many* *Scaling*	*Many-as-one* *Composite units* *Equivalent fractions*	*Spatial Visualisation* *Spatial Proportional Reasoning* **Added during cycle 2: Spatial Structuring*
Relationship between fraction ideas and spatial constructs	Similar to lesson six however children will use pattern blocks to manipulate and create fractional parts. A focus on operating on children's pattern block construction and establishing the relational proportions between the representations. E.g.: what would it look like if it were only half the size? Double the size? Three times etc.			
<u>Launch</u> Revisit/ modified task from Lesson 6: Ma wanted to buy a new tablecloth for the kitchen table. She asked Victoria and Sam to go to the shops and see if they could find one that was suitable. She asked for it to be in the colours of purple and orange, but this time, she wanted it to be ***more orange*** than purple. Victoria and Sam found the following tablecloths. Which one is ***more orange***? How much of the tablecloth is orange in each cloth? How do you know? Example of tablecloths				

Explore

Using the pattern blocks, children explore modelling 1 half, 1 quarter, a third and other fraction combinations of the choice, using different pattern blocks.

For example:
If this is a whole:



What is half? How many ways are there to represent half with the pattern clocks (connection to equivalent fractions)

If this is a third,  *what is a whole?*

Create a picture using different pattern blocks, focusing on where you can see fractions within your picture. Record and discuss your ideas with a friend.
What would your representation look like if it were half/ double, three times the size (area) of the original? If there are three triangles to half a hexagon, how many triangles in three hexagons? (Building up proto-ratio)

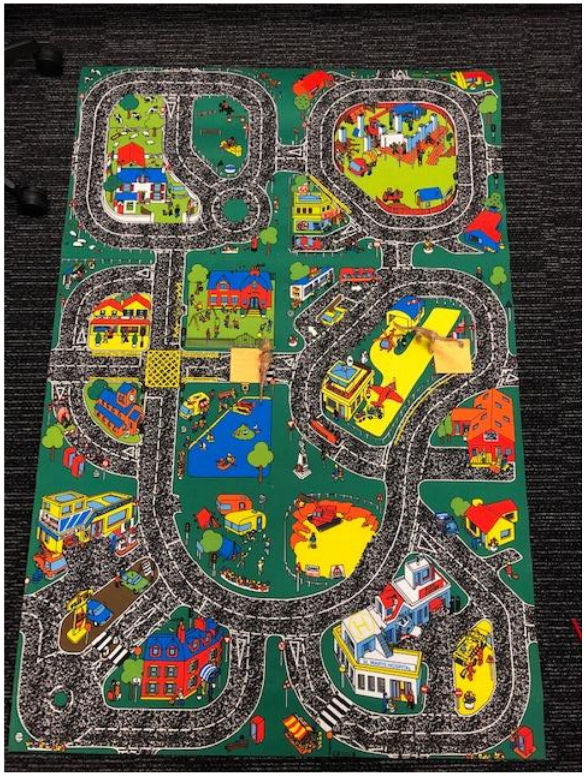
Summarise
Describe how you constructed your representations. What fractions did you create? What strategies did you use to help you? How did you shrink or enlarge your original picture? What patterns did you discover?

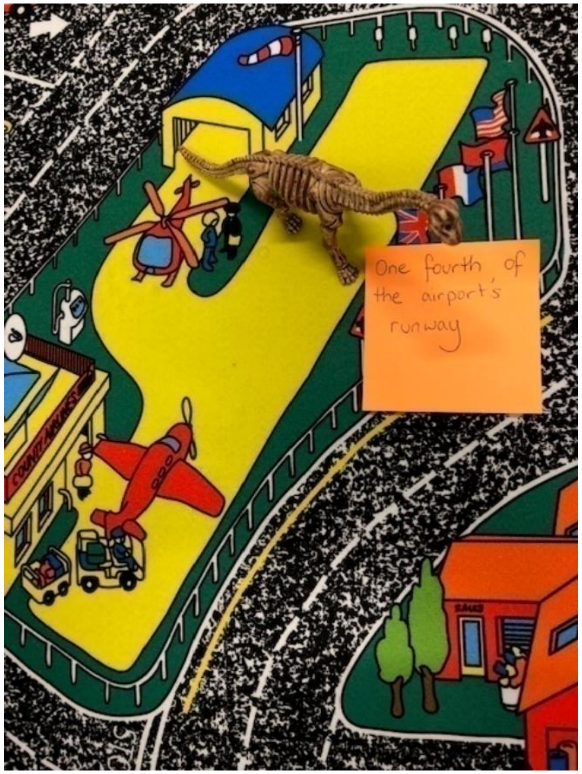
Lesson 8 The dinosaurs have escaped (Part 1)	**Rational Number Foci**			**Spatial Reasoning Foci**
	Fraction as Relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**
		Double/Halving *1-nth-of……* *Scaling* *Times as many*	*Composite units* *Unit fractions*	*Spatial Visualisation* *Mental Rotation* *Spatial proportional reasoning*
Relationship between fraction ideas and spatial constructs	Estimating fractional lengths of paths on carpet maps. Paths are not straight, so children need to engage in spatial visualisation and mental rotation to compare the length of multiple paths and use spatial proportional reasoning to estimate measures within a single pathway/ region.			

Launch
Introduce the new picture book “Knock, knock dinosaur” by Caryl Hart.

The dinosaurs have escaped the boys house! They’ve decided to explore the neighbourhood – here is the map. Somebody said they saw a T-Rex halfway between the boy’s house, and the zoo. Where would that be? (The Food store). Another person said they saw a dinosaur *halfway* between the central fountain and the duck pond – where would that be? (museum). One lady saw a velociraptor 2-thirds of the way along the road in front of the café, heading toward the food market…where would this dinosaur be?
Lots of scaffolding of directions/ position and then visualising 2-thirds of the length of said road.

Explore
You have taken a helicopter out to see if you can find some dinosaurs. On this map of the town (floor carpet), match the dinosaurs to their locations by sticking the correct tag on each dinosaur and placing on the mat (group task – each group will get a different mat).





Each set of directions will be specific to each mat, but include half, fourths, thirds, fifths and eights as well as language like twice as far etc. Children will also be encouraged to create their own fraction positions.

Summarise

Gallery walk: *Check out the other group's positions of their dinosaurs. Do you agree on their position? Is there another way of naming that position? What was hard about this task? What strategies did you team use to work out the position of your dinosaurs?*

Lesson 9 The dinosaurs have escaped (Part 2)	Rational Number Foci			Spatial Reasoning Foci
	Fraction as Relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**
		Doubling/ Halving *Scaling* *Times as many*	*Composite units* *Unit fractions* *Part-whole fractions*	*Spatial proportional reasoning*
Relationship between fraction ideas and spatial constructs	Estimating fractional lengths of paths on carpet maps. Paths are not straight, so children need to engage in spatial visualisation and mental rotation to compare the length of multiple paths and use spatial proportional reasoning to estimate measures within a single pathway/ region.			

Launch

Students start with a thin strip of paper, folding in half and to quarters.

Using this as a scaffold, a number line will be drawn on the board. Students will be invited to mark the halfway point on the number line, using their strips as a guide. Repeat with quarters. Scaffold "a dinosaur walking along the number line/ paper strip". If I walked from start of the number line to *here* (point to halfway, three quarters etc) how far would he have walked. A focus on measurement interpretation.

Repeat the introduction for the class from previous lesson. This time, pairs of students have this map between 2, and I call out clues where a dinosaur was found, and they need to mark on their maps.

The clues were: halfway between the central fountain an *a* park bench (they choose which park bench)

Three times the length of the school

1- quarter of the way through the zoo, from the front gate.

Explore

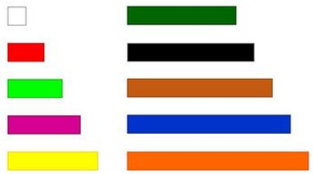
Each group of students gets a different map and set of instructions. They need to work as a team to find their dinosaurs. Then the children create at least one new instruction for another group to find.

Summarise

How did your group decide, and problem solve each of the clues? How do you know that your location is correct? Could there be other possibilities – why?
Gallery walk: *Check out the other group's positions of their dinosaurs. Do you agree on their position based on their task cards? Is there a different position the dinosaur could have been standing?* (i.e., one third of the runway depends on which end of the runway is considered the 'start').
What was hard about this task? What strategies did your team use to work out the position of your dinosaurs?

Lesson 10 The dinosaurs have escaped (Part 3)	**Rational Number Foci**			**Spatial Reasoning Foci**
	Fraction as Relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**
	Distribution	*Doubling/ Halving* *Scaling* *Times as many*	*Composite units* *Unit fractions* *Part-whole fractions*	*Spatial proportional reasoning (Scaling)*
Relationship between fraction ideas and spatial constructs	Creating scaled representations of fractional paths, naming and describing the distributions of the measures: half of …path is greater than half of another path.			

Launch



If this is one (orange, dark green etc) – what colour is half? (Spatial proportion activity)
If this is a 2 thirds (dark green) what is one whole?
What is 3 times light green? How many ways can you name it?
Give students a bag of rods in groups for them to explore.

Explore

Children represent parts of their carpet map that described where they saw each dinosaur, on an A3 sheet of paper. They need to draw the points of interest – e.g., the runway of the airport; draw the position of the dinosaur and then write in words their explanation such as, "dinosaur located *1 quarter* of the way along the runway".

Children can draw a map of another's whole (or part thereof) mat, again describing the position of the dinosaur in a fraction sense.

Summarise
*Intentional Teaching: *What is the same about the carpet map (zoo/ airport/ farm etc) and your map?* (Same proportions/ fraction, different scale etc).
What is different? (Absolute size).

Gallery walk: *Check out the other group's positions of their dinosaurs. Do you agree on their position based on their task cards? Is there a different position the dinosaur could have been standing?* (i.e., one third of the runway depends on which end of the runway is considered the 'start').

<table>
<tr><td rowspan="3">Lesson 11
How many steps?</td><td colspan="3">Rational Number Foci</td><td>Spatial Reasoning Foci</td></tr>
<tr><td>Fraction as Relation</td><td>Fraction as Operator</td><td>Fraction as Measure</td><td>Construct</td></tr>
<tr><td>Many-to-one
Distribution
Proto-ratio</td><td>Partitive division/ recursive multiplication
Times-as-many</td><td></td><td>Spatial Visualisation
*Spatial structuring</td></tr>
<tr><td>Relationship between fraction ideas and spatial constructs</td><td colspan="4">Children explore discrete part-part relations of footsteps, i.e., dinosaur: human quantities. Focus is on visualising the many-to-one relationship in the outset (i.e., for each Dino step, we take many steps to travel the <u>same</u> distance). Naming the relationship as times-as-many and building up/ building down to represent pro-ratios.</td></tr>
</table>

Launch
It's getting cold outside. Each dinosaur needs a pair of boots. For each dinosaur (2 legs or 4) how many boots? What if there were 5 dinosaurs? Think about how you can record this.

Explore
If one dinosaur step was 2 of your normal steps, how many of your steps would you need to take for 5 dinosaur steps? (Model this on the board/ with plastic counters)

Use materials to help you in any way you like.

If we take 3 steps to every dinosaur step, how many steps do we take in 2 dinosaur steps?
What if there are 5 dinosaur steps – how many steps will we need to take?
What if the dinosaur had taken 6 steps, and you had taken eighteen? What is the smallest number of steps you would need to take for 1 Dino step?

Summarise

What did you discover? How can we represent this in a way that helps us describe what is happening

Lesson 12 Animal Proportions	**Rational Number Foci**			**Spatial Reasoning Foci**
	Fraction as Relation	**Fraction as Operator**	**Fraction as Measure**	**Construct**
	Distribution *Equipartitioning* *multiple wholes*	*Doubling / Halving* *Times as Many* *1-nth-of...*	*Composite units* *Unit fractions*	*Spatial proportional reasoning*
Relationship between fraction ideas and spatial constructs	Exploring and preserving continuous part-part quantities and relationship when replicated (enlarged/ shrunk).			

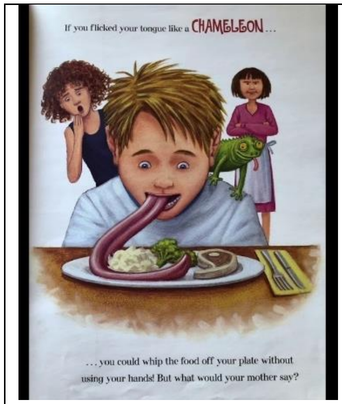
Launch

If you ate like a dinosaur you would eat A LOT! The t-rex has eaten through a heap of muesli and chocolate bars but hasn't quite finished each one before it started on the next.

Mum keeps a few boxes of muesli/ chocolate bars in the pantry for snacks. They are not all the same size. Which bar, did the T-rex eat the MOST of? How do you know? Which bar did it eat the LEAST of? How do you know?
(Students will receive 2 bars in various sizes with various amount eaten – as indicated by a different colour e.g., grey section uneaten, brown section eaten etc).

Explore

Some animals – not as big as dinosaurs, have some pretty interesting characteristics. For example, a Chameleon's tongue is twice the length of their bodies. (Show a picture of a chameleon from *If you hopped like a frog* (Schwartz, 1990) picture book).



If you were a chameleon – but the same height you are now, how long would your tongue be? Discuss strategies you would use to figure this out (i.e., model using freeze tape to measure heights and partition in half).

Using strips of paper, make your tongue based on the chameleon's proportions (twice the length of your body = length of tongue). What if the length of the Chameleon's tongue was only one third of the length of its body? How many times is your body the length of your tongue?

If you were a frog, you could hop five times the length of your leg. How far could you hop if you were a frog? Estimate first, think about what strategies you would use.

Provide students with different length of freeze tape: *If this was a frog jump, how long would its leg be?* How do you know? (Visualising fractions of a measure).

Summarise
What is the difference between one fifth and one third? Think about how you would describe these fractions to a friend.
Which is longer: A tongue that is a one third of your height, or a tongue that is one half of your height?
What do the names of the fraction parts tell us about the size of the part?
What strategies helped you?

<table>
<tr><td rowspan="3">Lesson 13
Feeding dinosaurs</td><td colspan="3">Rational Number Foci</td><td>Spatial Reasoning Foci</td></tr>
<tr><td>Fraction as Relation</td><td>Fraction as Operator</td><td>Fraction as Measure</td><td>Construct</td></tr>
<tr><td>Many-to-one
Distribution
Proto-ratio</td><td>Partitive division/ recursive multiplication
Times as many
1-nth-of...</td><td>Many-as-one</td><td>Spatial Visualisation

Added during cycle Two: Spatial Structuring</td></tr>
<tr><td>Relationship between fraction ideas and spatial constructs</td><td colspan="4">Children explore discrete part-part relations of pies and visualise the many-to-one and distribution of these quantities. Children build up and build down in pro-ratios to explore naming the quantities by changing the referent units (many-to-one versus many-as-one/ times as many/ 1-nth-of... etc).</td></tr>
</table>

Launch

Imagine you and your friend were sharing some chicken nuggets between you. However, you were STARVING, but your friend was not that hungry. You both decided that you should get twice as many nuggets as your friend. How many nuggets would you both get? Is there more than one solution?

Explore

Dinosaurs eat so much food! Did you know, that if you were a T-rex, you could eat 300 hamburgers in one mouthful!

Mum decides to feed the dinosaurs some of her pies she has made. For each dinosaur, she has made 3 pies. How many dinosaurs will she feed, with 18 pies? How can you use the strategies in the dinosaur step problem to help you with this?

How many dinosaurs will eat one half of the pies?

Summarise

Explain your solution to the problem and the strategies you used to solve. What are some different strategies the class used?

Appendix D: RMIT Ethics Approval

RMIT UNIVERSITY

College Human Ethics Advisory Network (CHEAN)
College of Design and Social Context (DSC)
NHMRC Code: EC00237

Notice of Approval

Date:	**21 January 2019**	
Project number:	**CHEAN A 21162-10/17**	
Project title:	**'Exploring the Efficacy of an Alternative Approach to the Teaching and Learning of Fractions in the Early Years of Primary School.'**	
Risk classification:	**Low Risk**	
Investigator(s):	**Emerita Professor Dianne Siemon, Mrs Chelsea Cutting, Dr Angela Rogers**	
Approval period:	From: **21 January 2019**	To: **21 January 2022**

I am pleased to advise that your amendment and extension request has been granted ethics approval by the Design and Social Context College Human Ethics Advisory Network (DSC CHEAN), as a sub-committee of the RMIT Human Research Ethics Committee (HREC). The CHEAN approves the change to the project title, and the change of year levels and number of classes involved in the research. Ethics approval is extended to 21 January 2022.

Terms of approval:

1. **Responsibilities of investigator**
   It is the responsibility of the above investigator/s to ensure that all other investigators and staff on a project are aware of the terms of approval and to ensure that the project is conducted as approved by the CHEAN. Approval is only valid whilst the investigator/s holds a position at RMIT University.
2. **Amendments**
   Approval must be sought from the CHEAN to amend any aspect of a project including approved documents. To apply for an amendment please use the 'Request for Amendment Form' that is available on the RMIT website. Amendments must not be implemented without first gaining approval from CHEAN.
3. **Adverse events**
   You should notify HREC immediately of any serious or unexpected adverse effects on participants or unforeseen events affecting the ethical acceptability of the project.
4. **Participant Information Sheet and Consent Form (PISCF)**
   The PISCF and any other material used to recruit and inform participants of the project must include the RMIT university logo. The PISCF must contain a complaints clause.
5. **Annual reports**
   Continued approval of this project is dependent on the submission of an annual report. This form can be located online on the human research ethics web page on the RMIT website.
6. **Final report**
   A final report must be provided at the conclusion of the project. CHEAN must be notified if the project is discontinued before the expected date of completion.

7. **Monitoring**
   Projects may be subject to an audit or any other form of monitoring by HREC at any time.
8. **Retention and storage of data**
   The investigator is responsible for the storage and retention of original data pertaining to a project for a minimum period of five years.

Please quote the project number and project title in any future correspondence.

On behalf of the DSC College Human Ethics Advisory Network, I wish you well in your research.

Dr David Blades
DSC CHEAN Secretary
RMIT University
dscethics@rmit.edu.au

Appendix E: Government of South Australia Ethics Approval

Government of South Australia
Department for Education

System Performance
31 Flinders Street
Adelaide SA 5000
GPO Box 1152
Adelaide SA 5001
DX 541
Tel. +61 8 8226-0809
Education.ResearchUnit@sa.gov.au
www.education.sa.gov.au

Reference No: 2018-0013

Ms Chelsea Cutting
[redacted]

Dear Ms Cutting

Your modification request relating to the research project titled *"Exploring the Efficacy of an Alternative Approach to the Teaching and Learning of Fractions in the Early Years of Primary School"* has now been reviewed by a senior Department for Education reviewer with respect to protection from harm, informed consent, confidentiality and suitability of arrangements.

Accordingly, I am pleased to advise you that your request to include younger children in the project for advancing knowledge about how to best teach fractions has been approved subject to the following conditions:

- A copy of any final reports, conference presentations or manuscripts accepted for publication is submitted to the Education.ResearchUnit@sa.gov.au mailbox 30 days prior to their publication, presentation or release.
- That the department is notified when findings are to be released to other government or nongovernment agencies or to participating sites.

Please contact Ms Betty Curzons from the Data Reporting and Analytics division on (08) 8226 0809 or email: Education.ResearchUnit@sa.gov.au for any other matters you may wish to discuss regarding the general review/approval process.

I wish you well with your research project.

[redacted]
Samuel Luddy
A/DIRECTOR, DATA REPORTING AND ANALYTICS

7 March 2019

Appendix F: Information Sheet

Participant Information Sheet

Title	Exploring the efficacy of an alternative approach to the teaching and learning of fractions in the early years of primary school.
Chief Investigator/Senior Supervisor	Emerita Professor Dianne Siemon
Associate Investigator(s)/Associate Supervisor(s)	Doctor Angela Rogers
Principal Research Student(s)	Chelsea Cutting

What does my participation involve?

1 Introduction

You are invited to take part in this research project, which is called '*Exploring the efficacy of an alternative approach to the teaching and learning of fractions in the early years of primary school*'. You have been invited because you are a South Australian Department for Education teacher in Years 1 or 2, who currently teaches mathematics. Your contact details were obtained from your school's general email address.

This Participant Information Sheet/Consent Form tells you about the research project. It explains the processes involved with taking part. Knowing what is involved will help you decide if you want to take part in the research.

Please read this information carefully. Ask questions about anything that you don't understand or want to know more about. Before deciding whether or not to take part, you might want to talk about it with a colleague or your Principal.

Participation in this research is voluntary. If you don't wish to take part, you don't have to.

If you decide you want to take part in the research project, you will be asked to sign the consent section. By signing it you are telling us that you:
- Understand what you have read
- Consent to take part in the research project

You will be given a copy of this Participant Information and Consent Form to keep.

2 What is the purpose of this research?

The aims of this research are to develop an evidence based, alternative approach to teaching fractions in the early years of primary school (Years 1 or 2).

Many primary students have significant misconceptions about fractions that hinder understanding and restrict the development of effective strategies for working with fractions (AAMT, 2017).

The development of an alternative approach is intended to assist and improve the teaching and learning of fractions in the early years of primary schooling. This will have immediate benefits to the participating teachers, students and their schools, however there is also the potential that the findings in this research may help improve and extend learning in other areas of rational number and proportional reasoning concepts.

This research is for the purpose of obtaining a degree through RMIT University.
The results of this research will be used by the researcher Chelsea Cutting to obtain a Doctor of Philosophy (PhD) degree.

3 What does participation in this research involve?

- You will be required to sign a consent form prior to your involvement in the research. This consent form includes gaining consent from your Principal to be part of this research project and the consent from the parents of the students in your class.
- Before the research commences, the researcher will observe your mathematics classes over a series of your normally scheduled mathematics classes, to understand the context of your class and students. You will also be individually interviewed by the researcher to gain insight into your teaching methods.
- You will <u>not be able</u> to teach your normal unit of work on fractions at all during 2019.
- In late term 2 or early term 3 2019, the researcher will teach a unit of work to your class over a period of 3-4 weeks, which utilises this evidence based alternative approach.
- Copies of student work samples and anecdotal notes may be taken, but no audio or visual recordings will be taken of the children.
- There are no costs associated with participating in this research project, nor will you be paid.
- You will receive a copy of all of your student's data and progress for your own assessment and reporting purposes.

4 Other relevant information about the research project

- The aim is to have at least 2 teachers participate in this the project from either Years 1 or 2.
- There is no requirement for teachers to teach, administer or assess any of the tasks; this will be done by the researcher.
- You will receive all assessment data from the unit of work the researcher has taught, for you own assessment and reporting purposes.

5 Do I have to take part in this research project?

Participation in any research project is voluntary. If you do not wish to take part, you do not have to. If you decide to take part and later change your mind, you are free to withdraw from the project at any stage.

If you do decide to take part, you will be given this Participant Information and Consent Form to sign and you will be given a copy to keep.

Your decision whether to take part or not to take part, or to take part and then withdraw, will not affect your relationship with the researchers or with RMIT University.

6 What are the possible benefits of taking part?

We cannot guarantee or promise that you will receive any benefits from this research; however, you may appreciate contributing to knowledge. Possible benefits may include:

- An understanding of the intuitive idea's students develop when learning fractions in the early years.
- An understanding of how a range of different strategies are utilised in the conceptual development of fractions.
- An understanding of the extent to which these strategies assist primary school aged children when learning fractions and how they can promote the proficiencies of understanding, reasoning, fluency and problem solving.
- A greater awareness of such strategies and the development of more effective pedagogical approaches to teaching fractions.
- Involvement in this project will count toward your 60 hours of Professional Learning required for your teacher registration.

7 What are the risks and disadvantages of taking part?

- Your participation will involve allowing the researcher to administer a 3–4-week unit of work during your normal mathematics lessons in term 2 or 3. Therefore, a disadvantage is that you cannot teach your fraction unit at all during 2019.

8 What if I withdraw from this research project?

If you do consent to participate, you may withdraw at any time. If you decide to withdraw from the project, please notify Chelsea Cutting, via email or phone:

You have the right to have any unprocessed data withdrawn and destroyed.

9 What happens when the research project ends?

The collected from the study will be analysed by the researcher. As this forms part of the researcher's PhD degree, the results (de-identified) will be published in the thesis at the completion of the degree, no later than 2023.

How is the research project being conducted?

10 What will happen to information about me?

- Names and contact information (email or phone numbers) will only be collected and used by the researcher to communicate with the participating teachers during the research project.
- They will be stored under password protected files on the researcher's PC. No other personal or identifiable information about the participating teachers or schools will be collected, stored or used in the research project's publications (whether they are published or not).
- The only personal information recorded about the students will be their age and current year level.
- All work-samples or written observations collected by the participating teachers for the project will be de-identified before the researcher obtains the data. Pseudonyms such as "Student A, Teacher 1" will be assigned to all participants to ensure confidentiality when research findings are presented either in aggregate form or as individual responses.
- All data including non-identifiable work samples, anecdotal notes and observations will be kept on the researcher's PC under password protected files. Hard copies of data will be kept in a locked filing cabinet only accessible by the researcher. Once hard copies have been scanned and saved into the appropriate password protected files, the originals will be destroyed via a secure document shredding company or returned to the participating teacher.
- All electronic files associated with this study (including the scanned copies of original work samples, consent forms etc) will be stored, password protected by the researcher in line with the Public Records Office of Victoria Standard (02/01) for at least 5 years. This data will also be backed up on RMIT's digital infrastructure through AARNET CloudStor+ which is also password protected and only accessible to the researcher.
- By signing this form, you are giving extended consent which means any of the unidentifiable data collected can be used in related, future research projects.
- By signing the consent form you consent to the research team collecting and using de-identified information from you for the research project. Any information obtained in connection with this research project that can identify you, such as the consent form, will remain confidential by the data storage means outlined above.

It is anticipated that the results of this research project will be published and/or presented in a variety of forums. In any publication and/or presentation, information will be provided in such a way that you cannot be identified, as all data relating to this study will be unidentifiable. No names of participating teachers, students and schools will ever be included in published or unpublished materials.

In accordance with relevant Australian and/or South Australian privacy and other relevant laws, you have the right to request access to the information about you that is collected and stored by the research team. You also have the right to request that any information with which you disagree be corrected. Please inform the research team member named at the end of this document if you would like to access your information.

Any information that you provide can be disclosed only if (1) it protects you or others from harm, (2) if specifically allowed by law, (3) you provide the researchers with written permission. Any information obtained for the purpose of this research project and for the future research that can identify you will be treated as confidential and securely stored.

11 Who is organising and funding the research?

This research project is being conducted by Chelsea Cutting as part of her PhD study.

12 Who has reviewed the research project?

All research in Australia involving humans is reviewed by an independent group of people called a Human Research Ethics Committee (HREC). This research project has been approved by the RMIT University HREC.
The Department for Education and Child Development South Australia have reviewed this project and provided ethics approval (forthcoming).
This project will be carried out according to the *National Statement on Ethical Conduct in Human Research* (2007). This statement has been developed to protect the interests of people who agree to participate in human research studies.

13 Further information and who to contact

If you want any further information concerning this project, you can contact the researcher – Chelsea Cutting on or any of the following people:

Research contact person

Name	Emerita Professor Dianne Siemon
Position	Senior supervisor
Telephone	[removed]
Email	

14 Complaints

Should you have any concerns or questions about this research project, which you do not wish to discuss with the researchers listed in this document, then you may contact:

Reviewing HREC name	RMIT University
HREC Secretary	
Telephone	
Email	human.ethics@rmit.edu.au
Mailing address	Research Ethics Co-ordinator Research Integrity Governance and Systems RMIT University GPO Box 2476 MELBOURNE VIC 3001

Appendix G: Consent Form

Consent Form

Title	**Exploring the efficacy of an alternative approach to the teaching and learning of fractions in the early years of primary school**
Senior Supervisor	Emerita Professor Dianne Siemon
Associate Supervisors	Dr Angela Rogers
Research Student	Chelsea Cutting

Acknowledgement by Participant

I have read and understood the Participant Information Sheet.

I understand the purposes, procedures and risks of the research described in the project.

I have had an opportunity to ask questions and I am satisfied with the answers I have received.

I freely agree to participate in this research project as described and understand that I am free to withdraw at any time during the project without affecting my relationship with RMIT.

I understand that I will be given a signed copy of this document to keep.

Name of Participant (please print) ______________________________

Signature ____________________ Date ____________________

Declaration by Researcher†

I have given a written explanation of the research project, its procedures and risks, and I believe that the participant has understood that explanation.

Name of Researcher† (please print) ______________________________

Signature ____________________ Date ____________________

† An appropriately qualified member of the research team must provide the explanation of, and information concerning, the research project.

Note: All parties signing the consent section must date their own signature.

Appendix H: Raw Scores from Pre and Post Task Based Interview

Raw Scores for TBI: Class B (n=23)

Set One: Trusting the Count				
Item	**Assessment Phase**	**No/Incorrect Response**	**Partially Correct Response**	**Correct Response**
1 Subitising cards	***Pre***	2	18	3
	Post	0	6	17
2 Hidden Counters	***Pre***	5	10	7
	Post	2	7	14
3 Tens Frame Bananas	***Pre***	7	5	11
	Post	2	9	12
4 Hidden Dots	***Pre***	15	5	3
	Post	8	11	4
Set Two: Place Value				
Item	**Assessment Phase**	**No/Incorrect Response**	**Partially Correct Response**	**Correct Response**
5 26 Counters	***Pre***	5	16	2
	Post	0	11	13
6 Place-Value Bundles	***Pre***	8	12	3
	Post	2	8	12
7 More than/ Less than…	***Pre***	18	5	0
	Post	9	9	6
8 Proportional Number line	***Pre***	11	8	4
	Post	2	10	11
9 Four-digit number task	***Pre***	19	4	0
	Post	16	5	2
Set Three: Fractions and Spatial Reasoning				
TBI Item	**Assessment Phase**	**No/Incorrect Response**	**Partially Correct Response**	**Correct Response**
10 Folding fractions	***Pre***	9	13	1
	Post	0	5	18

11 What fraction is green?	***Pre***	22	1	0
	Post	3	2	18
12 Comparing unit fractions	***Pre***	23	0	0
	Post	3	1	19
13 Cordial mixtures	***Pre***	20	3	0
	Post	5	4	14
14 Bags of wool	***Pre***	18	2	1
	Post	3	2	18
15 Missing faces	***Pre***	20	1	2
	Post	12	0	11
16 Halving the stars	***Pre***	9	7	7
	Post	1	0	22
17 Number flip	***Pre***	16	2	5
	Post	2	1	19
18 Gisele's paper square	***Pre***	9	2	12
	Post	2	0	21
19 Scale the picture	***Pre***	4	17	2
	Post	1	2	19
20 What's the object?	***Pre***	15	4	4
	Post	4	1	18
21 Eating pies	***Pre***	21	2	0
	Post	4	5	14
22 Fred's Pizza	***Pre***	23	0	0
	Post	11	0	12
23 Plant growth rate	***Pre***	18	1	4
	Post	8	0	15
24 How big is a half?	***Pre***	16	7	0
	Post	7	6	10

Raw Scores for TBI: Class C*
Pre-Intervention: n=21
Post-Intervention n= 15

***Only the data of the 15 children who completed both the pre and post TBI was used for the paired sample sign test.**

Set One: Trusting the Count				
TBI Item	**Assessment Phase**	**No/Incorrect Response**	**Partially Correct Response**	**Correct Response**
1 Subitising cards	***Pre (n=21)***	7	12	2
	Post (n=15)	0	10	5
2 Hidden counters task	***Pre (n=21)***	9	10	2
	Post (n=15)	2	9	4
3 Tens frame bananas	***Pre (n=21)***	8	9	4
	Post (n=15)	3	10	2
4 Hidden Dots task	***Pre (n=21)***	16	5	0
	Post (n=15)	7	8	0

Set Two: Place Value				
Item	**Assessment Phase**	**No/Incorrect Response**	**Partially Correct Response**	**Correct Response**
5 Counting 26 counters	***Pre (n=21)***	11	9	1
	Post (n=15)	7	4	4
6 Place-Value Bundles	***Pre (n=21)***	12	8	1
	Post (n=15)	2	10	3
7 More than/ Less than…	***Pre (n=21)***	12	8	1
	Post (n=15)	4	6	5
8 Proportional Number line task	***Pre (n=21)***	8	12	1
	Post (n=15)	3	5	7
9 Four-digit number task	***Pre (n=21)***	19	2	0
	Post (n=15)	11	3	1

Set Three: Fractions and Spatial Reasoning				
TBI Item	**Assessment Phase**	**No/Incorrect Response**	**Partially Correct Response**	**Correct Response**
10 Folding fractions	***Pre (n=21)***	3	16	2
	Post (n=15)	0	15	0
11 What fraction is green?	***Pre (n=21)***	15	6	0
	Post (n=15)	5	5	5
12 Comparing unit fractions	***Pre (n=21)***	21	0	0
	Post (n=15)	2	2	11
13 Cordial mixtures	***Pre (n=21)***	16	0	5
	Post (n=15)	6	1	8
14 Bags of wool	***Pre (n=21)***	12	4	4
	Post (n=15)	8	3	5
15 Missing faces	***Pre (n=21)***	20	1	0
	Post (n=15)	7	5	3
16 Halving the stars	***Pre (n=21)***	5	3	13
	Post (n=15)	4	4	7
17 Number flip	***Pre (n=21)***	12	0	9
	Post (n=15)	0	8	7
18 Gisele's paper square	***Pre (n=21)***	9	0	12
	Post (n=15)	0	4	11
19 Scale the picture	***Pre (n=21)***	4	15	2
	Post (n=15)	0	13	2
20 What's the object?	***Pre (n=21)***	12	2	7
	Post (n=15)	4	4	7
21 Eating pies	***Pre (n=21)***	21	0	0

	Post (n=15)	3	5	7
22 Fred’s Pizza	***Pre (n=21)***	20	0	1
	Post (n=15)	3	2	10
23 Plant growth rate	***Pre (n=21)***	20	0	1
	Post (n=15)	11	1	4
24 How big is a half?	***Pre (n=21)***	17	4	0
	Post (n=15)	3	8	4