# Exact Travelling Wave Solutions in MHD and Plasma Physics

A thesis submitted in fulfillment of the requirements for the degree of

Doctor of Philosophy

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# Declaration

I certify that except where due acknowledgement has been made, the work is that of the author alone; the work has not been submitted previously, in whole or in part, to qualify for any other academic award; the content of the thesis is the result of work which has been carried out since the official commencement date of the approved research program; any editorial work, paid or unpaid, carried out by a third party is acknowledged; and, ethics procedures and guidelines have been followed.

***Musaad Sabih Aldhabani***

**07 AUGUST 2020**

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# SUMMARY

Over the last four decades, much attention has been paid to nonlinear studies on exact travelling wave solutions for some nonlinear physical systems such as ion-acoustic waves, incompressible magnetohydrodynamics (MHD) waves, and isothermal magnetostatic atmospheres. It is not surprising then that exact travelling solution-based techniques have become important in the study of nonlinear physical systems.

Nonlinear evolution equations are commonly used as models for explaining complex physical processes in different fields of sciences, especially in fluid mechanics, solid state physics, plasma physics and chemical physics. Given a nonlinear partial differential equation (NLPDE), there is no general way of knowing whether it has soliton solutions or not, or how the soliton solutions can be found. In order to gain a better understanding of the underlying phenomena as well as their further applications in practical life, it is important to seek their exact solutions.

Analytical solutions to NLPDEs play an important role in nonlinear science, especially in nonlinear physical science since they can provide more insight into the physical aspects of the problem and thus lead to further

applications. Moreover, new exact solutions may help researchers to find new phenomena [1-11]. In various science and engineering fields, nonlinear wave phenomena arise, such as in fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical physics and geochemistry. In the theory of nonlinear wave dispersion, dissipation, diffusion, reaction and convection, nonlinear wave phenomena are very common [12-22].

This thesis consists of a summary and seven chapters as detailed below.

Chapters 1, 2 and 3 are Introductory chapters dealing with the historical discussions of solitons and travelling wave solutions. This is followed by a brief survey of how mathematicians and physicists began to work on certain problems of mutual interest. This is followed by a brief survey of MHD and ion-acoustic waves. In addition these chapters provide the background material used in this thesis. They covers the fundamental concepts of known results concerning our works to make this thesis somewhat self-contained.

Chapters 4,5 and 6 are primarily focused on finding travelling wave solutions in certain plasma physics problems. In chapter 4 we address acoustic waves of ions, which are the variation of sound waves in plasma physics. Such waves spread as more mobile electrons protect the electrical field from the ions and set up their own field to move the ions in a tenuous plasma. These waves could be excited, but there are infrequent collisions between the electrons and ions. In these waves the natural magnetic field is zero. However, the solutions in chapter 5 for a three-dimensional incompressible MHD system are obtained using the sine – cosine method and the Riccati auxiliary equation. This chapter obtains soliton solutions with the aid of the software Mathematica.

In chapter 6, the Bäcklund Transformations (BTs) method is proposed to look for exact solutions for the nonlinear differential equations resulting from solar MHD. The BTs methodology is applicable to search for exact solutions from magnetostatic equations in solar physics to the sinh-Poisson equations.

Finally, chapter 7 provides a brief conclusion and considerations for possible future work in these areas.

# Chapter 1
# Introduction

## 1.1 Overview

This thesis provides a brief survey of how mathematicians and physicists discovered and began working on some mutually interesting issues. The origin of non-linear differential partial equations (PDEs) is very old, but many advances happened in the latter half of the 20th century. One of the key reasons for researching non-linear PDEs was to explore propagation issues in non-linear waves. Similar problems exist in various fields of applied mathematics, engineering and fluid dynamics, non-linear effects in optics, solid mechanics, plasma physics and the quantum field theory.

Nonlinear wave equations generally given rise to numerous types of new solutions which differ considerably from solutions generated from linear approximations. Shock waves, heat waves and discrete waves are the best-known examples.

Nonlinear waves and solitons have, however, undergone a revolution in recent decades. Many extraordinary and surprising events during this transition were also noticed in natural, chemical, and biological processes.

There were many significant achievements including the discovery of soliton interactions, the inverse dispersion method to find effective exact solutions for several PDEs, and the evaluation of asymptotic interference in the study of non-linear evolution equations [23-27]. Equations of nonlinear design are often used as models in various fields of science to assist in the understanding of complex physical processes.

There is no general way of knowing, given a nonlinear PDE, whether soliton solutions exist and if they should be considered. It is essential to look at their exact solutions to understand the underlying phenomena and their further applications in practical life. Analytical solutions for nonlinear PDEs play an important role in nonlinear science, particularly in nonlinear plasma physics, because they provide a lot of physical insights that could lead to further applications. Indeed, new approaches can help investigators find new phenomena. Exact solutions of nonlinear PDEs, if available, allow a verification of results arising from numerical simulations and help to analyze solution stability [28-35].

In computational biology, the impulses visible in nerve fibers are seen as moveable waves [37]. Studies of ion-acoustic waves with superthermal particle distributions are critical for the understanding of space plasma observations.

The studies mentioned in the previous section focused primarily on the distribution of Maxwellian electrons or distributions of the Cairns type. In the case of superthermal plasmas, there have been a few studies that have attempted to analyse ion-acoustic solitary structures. Chuang and Hau [21] studied the dynamics of low amplitude ion-acoustic solitons. Saini et al. [84] researched the characteristics of the life for two-component ion-acoustic solitons.

Saini and Kourakis [83] investigated the presence of arbitrary ion-acoustic solitary waves in an unmagnetized plasma made up of ions and excess superthermal electrons (modelled by a kappa-type distribution).

Sultana et al. [102] recently studied magnetised ion-acoustic solitary waves in a two component plasma, with kappa-distributed electrons and fluid-cool ions. They used Sagdeev's pseudopotential approach under a quasi-neutrality condition to study the effects of solitary ion-acoustic waves on the obliqueness and superthermality of electrons. By studying obliquely propagating linear and nonlinear ion-acoustic waves in an electron-ion magnetised plasma [41-50], the influence of superthermal electrons modelled by the Lorentzian velocity distribution was examined.

They also used the theory of small amplitude waves to study ion-acoustic nonlinear waves. Nevertheless, in each of the aforementioned studies the impact of finite ion temperature on solitary structures was not considered. We can observe magnetic fields in nature as well as in industrial processes that influence the behaviour of fluids and flows. Magnetic fields are used for stirring, pumping, levitating, and heating liquid metals in the metallurgical industry.

The earth's magnetic field is created by the movement of the liquid core of the earth, shielding the ground from deadly radiation. The solar magnetic field produces sunspots and solar flares while the galactic magnetic fields drives star formation from interstellar gas clouds produce sunspots and solar flares. As a result of massive interactions between galaxies and the intracluster gas, these intracluster magnetic fields are likely generated by turbulent gas motion. They directly impact thermal conduction in galaxy clusters and consequently their evolution, but these are reviewed elsewhere (e.g. Saini et al. 2009; Saini and Kourakis 2010) [83,84].

In all these cases, the phenomena are investigated using the MHD approximation, where there is an electrically conductive or non- magnetic substance, such as a liquid metal or a hot ionized gase (plasma), or solid electrolytes [33-40].

### 1.2 Motivations and Research questions

The aim of this thesis is the investigation of travelling wave solutions for nonlinear partial differential equations that play an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In the past several decades, new exact solutions may help to explore new phenomena.

In this thesis, the natural questions to ask are as follows:

RQ 1: What is the significance of the research?

RQ 2: What is the travelling wave solution?

RQ 3: What methods are used to solve travelling wave problems?

RQ 4: What are the results in this research by using this method?

RQ 5: What are the different types of travelling wave solutions?

RQ 6: How to find new exact travelling wave solutions of the nonlinear equations arising in physical models?

RQ 7: What are the physical models and their importance?

### 1.3 Contributions and objectives of the research

One of the cornerstones in the study of both linear and nonlinear PDEs is wave propagation. A wave is a recognizable signal which is transferred from one part of the medium to another part with a recognizable speed of propagation. Energy is often transferred as the wave propagates, but matter may not be. We mention here a few areas where wave propagation is of fundamental importance.

- Fluid mechanics (water waves, aerodynamics)
- Acoustics (sound waves in air and liquids)
- Elasticity (stress waves, earthquakes)
- Electromagnetic theory (optics, electromagnetic waves, magnetohydrodynamics (MHD))
- Biology (epizootic waves)

- Chemistry (combustion and detonation waves)

In this research, a brief description of nonlinear phenomena and a survey on some nonlinear models is presented. We also discuss solitons solutions and various other types of travelling wave solutions. A brief account of some important and widely used analytic methods to obtain exact solutions of a variety of nonlinear PDEs relevant to physical problems is also given.

In this thesis we will find exact soliton solutions to some higher order nonlinear physical model equations. The obtained solutions with arbitrary parameters may be significant. The obtained solutions may be useful help in understanding the mechanisms behind the complicated nonlinear physical phenomena which are related to wave propagation in higher order nonlinear physical model equations.

Generally, nonlinear physical models are difficult to solve. So for the last few decades, a great deal of attention has been directed towards the solution (both exact and numerical) of these problems. During this project, our key research objectives will be:

The important mathematical preliminaries and the general basic consideration for the models considered in this thesis.

This work explains the basis of travelling wave solutions and develops a method to obtain analytical solutions for nonlinear physical models. We then find exact travelling wave solutions for some nonlinear physical models in

(a) Three dimensional incompressible MHD equations.

(b) Isothermal magnetostatic atmospheres equations.

(c) Some important equations of ion-acoustic waves.

The thesis consists of a summary and seven chapters, organized as follows:

**Chapter 1: Introduction:**

The introduction includes a short historical discussion of the early solitons ' ideas on travelling wave solutions, followed by a brief survey of how mathematicians and physicists noticed and began to work on certain problems of mutual interest. This is followed by a brief survey of MHD and ion-acoustic waves.

**Chapter 2: literature review**

This chapter consist of a short historical discussion of the early solitons ' ideas on travelling wave solutions, followed by a brief survey of how mathematicians and physicists noticed and began to work on certain problems of mutual interest. This is followed by a brief survey of MHD and ion-acoustic waves.

**Chapter 3: Equations of motion**

This chapter provides the background material used in this thesis. It covers the fundamental concepts of known results concerning our work to make this thesis somewhat self-contained.

**Chapter 4: Ion acoustic waves in plasma and soliton solutions**

In this chapter we address acoustic waves of ions, which are the variation of sound waves in plasma physics. Such waves spread as more mobile electrons protect the electrical field from the ion and set up their own field to move the ions in a tenuous plasma. Such waves could be excited, but there are infrequent collisions between the electrons and ions. In these waves the natural magnetic field is zero.

The equation of the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) were derived and analytically examined. The main components of KdV and mKdV solitons have been analyzed. It has been observed that the plasma system being considered supports the propagation of solitons obtained from the solutions of KdV and mKdV equations. We point out here that the results of this chapter are published in the Second International Conference of Mathematics, Statistics and Information Technology that was held in the Faculty of Science, Tanta University, December, 18 -20, 2018.

**Chapter 5: Travelling wave solutions for three-dimensional incompressible MHD equations**

In this chapter solutions for a three-dimensional incompressible MHD system are obtained using the sine – cosine method and the Riccati auxiliary equation. This chapter obtains soliton solutions with the aid of the software Mathematica. We point out here that the results of this chapter are published in the Journal of Applied Mathematics and Physics, 6(2018)114-121.

**Chapter 6: Analytical solutions for isothermal magnetostatic atmospheres equations**

In this chapter, the Bäcklund Transformations (BTs) method is proposed to look for exact solutions for the nonlinear differential equations resulting from Solar MHD. The BTs methodology is applicable to search for exact solutions from magnetostatic equations in solar physics to the sinh-Poisson equations. Under a gravitational field, the equations of magnetohydrostatic equilibria for plasma are also investigated analytically. The analysis of a family of isothermal magnetostatic atmospheres is carried out in a plane geometry with one ignorable coordinate corresponding to a uniform gravitational field. The distributed current $\boldsymbol{J}$ is guided along the $x-$axis where $x$ is the negligible horizontal coordinate. for the magnetic vector potential $u$ These equations transform into a single nonlinear elliptic equation. This equation depends on a given arbitrary function of $u$.

We point out here that the results of this chapter have been submitted to (Journal of Computational and Applied Mathematics).

**Chapter 7: Conclusion and further research**

This chapter provides a brief conclusion and considerations for possible future work in these areas.

# Chapter 2
# Literature Review

Nonlinear partial differential equations are used to model a large number of problems in physics, mathematics, and engineering. Analyzing the exact solutions to these nonlinear equations plays a very important role in the theory of solitons since they provide much knowledge about the physical system in question.

Various effective methods for constructing precise travelling wave solutions for nonlinear partial differential equations can be used. Such methods include the inverse scattering transformation [4]; the Bäcklund transformation [87-91]; the Darboux transformation [4]; the Hirota bilinear method [5]; the tanh-function method [65,66]; the sine-cosine method [8]; the expansion method [9]; the generalized Riccati equation [10]; the homogeneous equilibrium method [11]; the first integral method [12,13]; and the $\mathbf{G}^{/}/\mathbf{G}$ method of expansion [14-19].

Due to the availability of symbolic computation packages such as Maple and Mathematica, direct methods for building accurate NLPDE solutions have become increasingly attractive in recent years as all the tedious and complex calculations can be performed by a computer. This thesis aims to use a powerful method called the sine-cosine method and Bäcklund transform method to obtain exact bright, dark and cnoidal wave soliton solutions to some higher-order nonlinear equations modelling different physical phenomena [30-40].

Many structures in solid mechanics are typically modelled as standing waves [8], so determining the dynamics of those solutions is important. On a similar note, the significance of evaluating the KdV and mKdV equations within this project emerge from a common principle. The travelling wave solution is obtained when the model corresponding to the physical system is resolved.

These models generally take the form of partial differential equations (PDEs) in which the dynamics of the systems are exposed when solutions are found. Such solutions shall be expressed as $u(x;\, t) = U(z)$, -where $z = x - ct$. as $x$ and $t$ are spatial and time respectively with the wave speed given as $c$.

Actually, we could classify the travelling waves into forms due to certain characteristics (Figure 2.1). A travelling wave that approaches a constant state is given by $U(-\infty) = u_l$ and $U(\infty) = u_r$, with $u_l \neq u_r$. The resulting wave is known as a pulse wave if the constant states $u_l = u_r$, are identical. If a wave shows regularity, it is considered a spatially periodic wave $U(z + F) = U(z)$ where $F > 0$ [92-112].

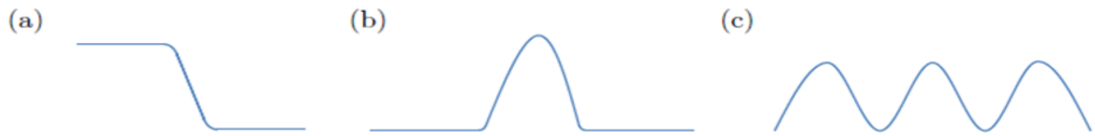
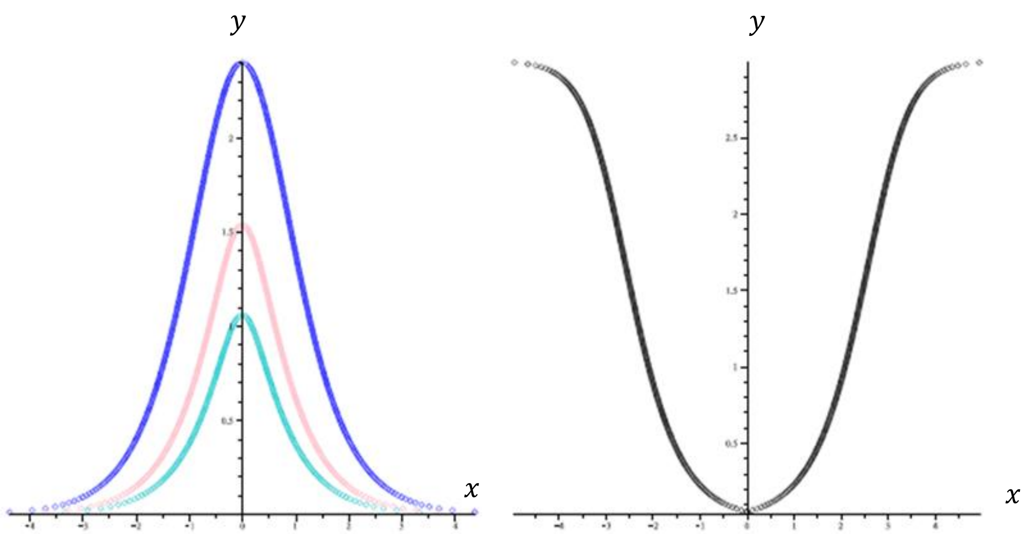
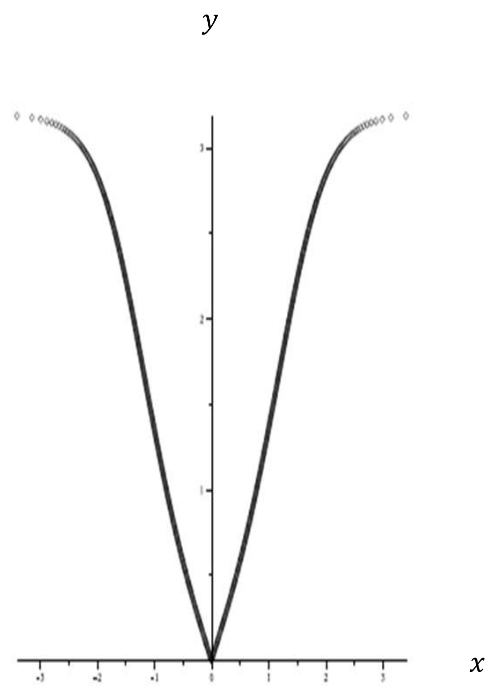



Figure 2.1: Forms of travelling waves: (a) wave front, (b) pulse and (c) spatially periodic travelling wave.




(a) Solitary wave of bell type (b) Solitary wave of valley type



(c) Peakon wave with valley type

Figure 2.2: Different kinds of solitary wave solutions

Plasma, also referred to as the "fourth state of matter," is an ionised gas that some claim to be the main constituents of 99 per cent of our visible universe. The stars, inter-galactic medium, nebulae and interstellar medium are all made up of plasma. The polar auroras, lightning and ionosphere are some examples near to our Sun.

A plasma that is a combination of neutral atoms, respectively positively and negatively ions, is a quasi-neutral liquid that is electrically conductive. Plasma can usually be achieved with electrical and magnetic fields due to these properties, resulting in a wide range of waves and oscillations in the acoustic, radio and optical spectrum.

For dispersive and nonlinear plasmas, nonlinearity can often compensate for the Dispersal Effects of a system, resulting in waves that can retain their original shape over large distances. Such stable waves [111-122] are called solitons. Typical examples are waves in shallow water and plasma waves, ion-acoustic waves. Both types of waves are governed by a common nonlinear wave equation named KdV equation [1-17].

Solitary waves act as particles just before they interact with one another and. because of their particle-like existence and their ability to maintain their identity over long distances, solitons are still a matter of study even today. Solitons can travel thousands of kilometres in optical fibres and can be used to transmit information. Chapter 2 explains in detail the background history of solitons and the fundamentals of solitons.

The theory behind soliton propagation within an inhomogeneous plasma is well known, particularly when the electrons are considered to be isothermal and are defined by a Boltzmann distribution. When the density is inhomogeneous, the soliton's properties are modified.

Existing theory does not explain soliton behavior when two-temperature electrons are present, or when the electrons are non-isothermal. Plasmas of trapped electrons may also be present in the auroral magnetosphere [132-136] where the presence of hot and cold electrons may be caused by heating and injection.

In space for example [3], the two-temperature electrons can be found in the E rings of Saturn. They are also present in the laboratory where, due to an external source and preferential heating of some electrons, both fast and slow components of electrons can occur.

The findings from our theoretical analyses may also be relevant to the understanding of particle and field data received from various spacecraft missions in the Earth's auroral ionosphere and magnetosphere.

Within galaxies, and indeed in many other astrophysical settings, the gas is partially or fully ionised and can carry electric currents which in turn generate magnetic fields. In general, the associated Lorentz force exerted on the ionised gas can no longer be neglected in the momentum equation for the gas.

Magneto-hydrodynamics (MHD) is the research of the magnetic field interaction and the plasma viewed as a fluid. In MHD we integrate Maxwell's electrodynamic equations with the fluid equations, including the Lorentz forces which arise from the electromagnetic field.

In uniform plasma [41-47], there are three categories of MHD waves :

- Fast magnetoacoustic wave: Quasi-isotropic wave caused by plasma and magnetic pressure forces in concert.

- Alfvén wave: Transverse (shear) perturbations following magnetic field, caused by magnetic tension force.

- Slow magnetoacoustic wave: Anisotropic wave caused by pressure forces in opposition.

Magnetohydrodynamic waves can becomes nonlinear when their amplitudes become significant. Some of the MHD waves observed a in solar flares and in the solar wind are nonlinear.

The waves of Alfvén provide an exact solution for the nonlinear MHD equations. Nonlinearity often results in wave amplitude development and steepening, just as an ocean wave undergoes these changes when it approaches the shoreline.

Shock waves are nonlinear waves with profiles that, in the ideal case, have particularly sharp discontinuities and over which dissipative processes balance any nonlinearities. Conservation relationships (mass, total energy, momentum, etc.) are often used to assess a shock wave 's character [61-80]. Taking into account the presence of both slow and quick magnetoacoustic waves, slow shocks and quick shocks may occur. In photospheric flux tubes and in the solar wind, slow shocks can occur and are also invoked in theories of magnetic reconnection. Fast shocks may be present in observed blast waves produced by ~~a~~ solar flare. Theoretical grounds also exist for believing that solitons can occur in the Sun [58-75].

# Chapter 3
# Equations of motion

This chapter presents some needed mathematics in the remaining chapters as well as some basic definitions of terms which will be used in this thesis.

## 3.1 Some useful definitions

**Definition 3.1.1: Wave**

A wave is a physical phenomenon defined by wavelength, frequency and amplitude. Also wave is a physical disturbance that can propagate through a medium and transmit energy from one place to another.

**Definition 3.1.2: Dispersive wave**

A propagating wave, satisfying a dispersion relation $\omega = \omega(n)$, such that $\omega/n$ is not a constant, where $\omega$ is the frequency and $n$ is the wave number, is called a dispersive wave. A dispersive wave is said to be nonlinear if the dispersion relation also depends on the amplitude of the wave, otherwise it is a linear dispersive wave.

**Definition 3.1.3: Non-dispersive wave**

Any propagating wave where the propagation speed is independent of the wavenumber, resulting in a wave that maintains its form.

**Definition 3.1.4: Solitary wave**

A localised nonlinear wave that travels with no change in shape and speed.

**Definition 3.1.5: Soliton**

A solitary wave which even after a collision with another solitary wave retains its identity.

**Definition 3.1.6: Dissipation**

A wave that loses amplitude is considered a dissipative wave due to the loss of energy over time.

**Definition 3.1.7: Travelling wave**

A travelling wave is a permanent type of wave that travels at a constant speed. Travelling wave solutions are typically obtained by reducing the nonlinear evolution equations to common ordinary differential equation associated with these. This is mainly done while using the ansatz $u(x,t) = u(\xi)$, where $\xi = x - kt$ and $k$ is the wave speed. This transforms the

PDE in $x,\ t$ to an ordinary differential equation in $\xi$ that could be solved by several suitable methods [1-11].

**Definition 3.1.8: Partial differential equation**

A partial differential equation is a differential equation that contains unknown multivariable functions and their partial derivatives. Partial differential equations are used to formulate problems involving functions of several variables. For example, $\frac{d^2 x}{d\ t^2} + t^2\ \frac{d\,x}{dt} + 2x = 0$ is an ordinary differential equation, while $\frac{\partial^2 u}{\partial\, x^2} + \frac{\partial^2 u}{\partial\, y^2} = 0$, is a partial differential equation.

**Definition 3.1.9: Linear and nonlinear partial differential equations**:

Partial differential equations are graded as linear or nonlinear. If there are no products or powers between the dependent variable and its derivatives, then the partial differential equation is considered linear. Anything else would be called nonlinear. For example, the most general first-order linear PDE for $u(x,t)$ would be $a(x,t)\frac{\partial\, u}{\partial\, t} + b(x,t)\frac{\partial\, u}{\partial\, x} + c(x,t)u = d(x,t)$, while $\frac{\partial\, u}{\partial\, t} + v(x,t)\frac{\partial^2 u}{\partial\, x^2} + u\frac{\partial u}{\partial\, x} = 0$, is a nonlinear partial differential equation.

**Definition 3.1.10: Nonlinear evolution equation (NLEEs)**

Typically, NLEEs are nonlinear partial differential equations with time $t$ being one of the independent variables. NLEE usually arise from the study of various physical structures such as water waves, plasma mechanics, harmonic lattices, and elastic rods.

## 3.2 Solitons

As per preliminary description, a soliton is a single, well-defined non-dissipative wave that can travel long distances without any change in size or shape [1-3]. Solitary waves are localized waves that, given the dispersion and nonlinearity of the medium, maintain their shapes.

One of the most startling features of nonlinear phenomena is the localised large-amplitude waves called solitons, which propagate without spreading and have particle-like properties. They have unique properties [12-19] as follows:

• Nonlinear distributed structures moving with permanent form and constant speed.

• They can cross each other and preserve their identities after a pair-wise collision.

Solitons are distinctive types of solitary waves. The vast group of nonlinear phenomena, commonly arising in various fields of applied mathematical sciences and physics such as condensed matter, plasma physique, turbulence theory, ocean dynamics, biophysics, fluid dynamics and star formation, solid-state physics, optical fibres, chemical kinestics and star-formation, have played an important role in science since the discovery of isolation waves known as solitons (1965) by Zabusky and Kruskal [132].

### 3.2.1 Formation of solitons

Solitons are created by a delicate balance between a medium's nonlinearity and dispersion. In particular, linear waves in dispersive media are scattered and distributed over great distances because the phase speed is $k$ −dependent.

In most nonlinear media, nonlinear effects begin to occur when the amplitude of a wave is not high. Nonlinearities can also increase the steepness of a wave front, leading to the breaking of the wave. Figure 3.2.a displays the dispersion of a waveform, while figure 3.2.b displays the steepening of the waveform induced by a nonlinearity. Figure 3.2.c demonstrates how a stable structure called a soliton is formed by the combination of these two results.

Although a plasma generally behaves like a nonlinear medium and almost all plasma waves are dispersed, solitons solutions are not necessarily present. However, nonlinear ion-acoustic waves can exhibit soliton behaviour and can be considered a typical example of plasma solitons. The linear dispersion relationship for ion-acoustic waves is obtained and the ion-acoustic velocity is discussed for the limiting cases of extremely dense or dilute systems [94-99].

The correct Korteweg-de Vries equation [49] for weakly nonlinear solutions is obtained using the reductive perturbation process, and the resulting propagation of the soliton is analysed. Depending on the value of the quantum parameter for the degenerate electrons, which influence the phase velocities in the dispersive medium, it is found that structures are created by the soliton hump and dip

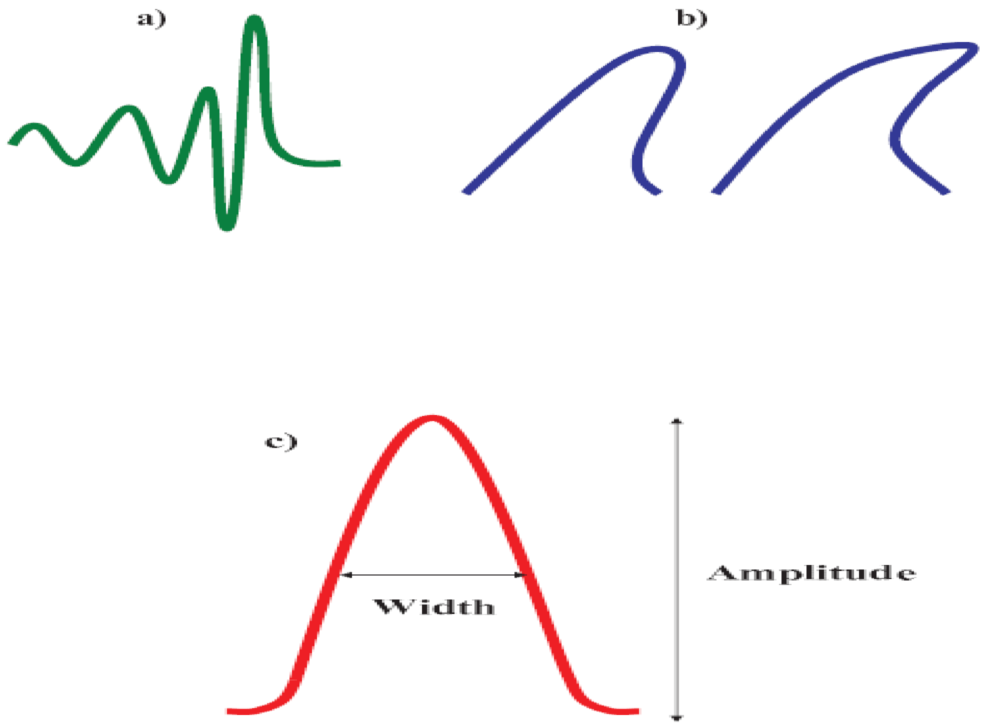



Figure 3.2: Formation of a soliton.

a) Linear dispersive waves have phase speeds ($c = \omega/k$ where $\omega$ is angular frequency and $k$ is wavenumber) that depend on their wavelengths.

b) Nonlinear wave: with the rise in amplitude, the wave steepens and has a tendency to break.

c) A Soliton.

### 3.2.2 The discovery of solitons and the KdV equation

Nonlinear science is fundamentally interdisciplinary and has an influence on mathematics, traditional sciences and engineering, as well as on the social sciences, especially economics and demographics. Along with Fractals and Chaos, the theory of solitons is recognized as an important branch of nonlinear science and has evolved rapidly in recent decades.

The theory of solitons is an important area in mathematical physics and applied mathematics that has grown rapidly since the sixties. Solitons act as particles as well as waves, and sometimes occur in nature. There are many important problems related to soliton theory in research fields such as nonlinear optics, plasma physics, fluid mechanics, quantum field theory and classical etc. The concept of a "soliton" has also become more widely understood over the last years [1].

Subsequently, many other scientists started researching this phenomenon, including Stokes, Airy, Boussinesq, and Rayleigh. Boussinesq and Rayleigh both obtained the solitary wave's approximate properties among these authors, and Boussinesq derived an equation for the one-dimensional propagation of solitons and found an analytical solution.

In 1895, while observing the journey of shallow water waves produced in a shallow stream, D. Korteweg and G. De Vries [49] obtained a new nonlinear one-dimensional formula describing these particular types of water waves. This equation is now known as the KdV equation [55-66],

$$u_t + \alpha\, u\, u_x + u_{xxx} = 0. \tag{3.1}$$

In 1965, Zabusky and Kruskal [132] discovered some remarkable outcomes when studying the non-linear mechanism of collisions between solitary waves in a plasma. The results of their experiments were quite surprising after some numerical simulations on the computer.

Soliton theory, however, offered some approaches for dealing with nonlinear problems. In particular, the inverse scattering method can be called the Fourier method for nonlinear problems in some context. In addition to the inverse dispersion approach, there are plenty of elegant and efficient methods to create exact solutions.

Many mathematics branches like Classical and Functional Analysis, Lie Groups, Lie Algebras, Differential Geometry, Algebraic Geometry, Topology, Dynamic Systems, and Computational Mathematics are important tools for soliton research. On the other hand, the study of solitons also promotes the development of these areas. For these reasons, both mathematicians and physicists pay much attention to soliton theory. This is a very active area of study and it encompasses an increasing variety of subjects. Several conferences on this field were held in each of the last ten years. A number of books have been published, and there are several articles in different journals on soliton theory.

A variety of methods have been developed to solve these equations which are now commonly used, such as the reverse scattering transformation, Lie symmetry analysis, Darboux transformation, Bäcklund transformation, Hirota bilinear method, and the Wronskian determinant method [38-48]. Here we define the main characteristics of some of the widely used direct methods for solving nonlinear PDEs that are also used in this work.

### 3.2.3 A sine–cosine method

1. The wave variable $\xi = x - ct$ where $c$ is a constant, transforms a nonlinear PDE in two independent variables $(x, t)$

$$P(u, u_t, u_x, u_{xx}, u_{3x}, \dots \dots) = 0, \tag{3.2}$$

where $u(x, t)$ is the travelling wave solution to a nonlinear ODE

$$Q(u, u^{/}, u^{//}, u^{///}, \cdots) = 0. \tag{3.3}$$

Notice that

$$\frac{\partial}{\partial t} = -c\,\frac{d}{d\xi}, \quad \frac{\partial^2}{\partial t^2} = c^2\,\frac{d^2}{d\xi^2}, \quad \frac{\partial}{\partial x} = \frac{d}{d\xi}, \qquad \frac{\partial^n}{\partial x^n} = \frac{d^n}{d\xi^n}. \tag{3.4}$$

Eq. (3.3) can then be integrated as long as all the terms encompass derivatives.

2. The answers to many nonlinear equations can be stated in the cosine form

$$u(x,t) = \lambda\ cos^{\,m}(\mu\,\xi), \qquad |\xi| \leq \frac{\pi}{2\,\mu} \tag{3.5}$$

or in the sine form

$$u(x,t) = \lambda\ sin^{\,m}(\mu\,\xi), \qquad |\xi| \leq \frac{\pi}{\mu} \tag{3.6}$$

where $\lambda$ and $m$ are parameters that will be determined, $\mu$ and $c$ are the wave number and the wave speed respectively. We then utilise

$$u(\xi) = \lambda\ cos^{\,m}(\mu\,\xi), \tag{3.7}$$

to get

$$u^{/}(\xi) = -\ \lambda\,\mu\,m\ cos^{\,m-1}(\mu\,\xi)\sin(\mu\xi), \tag{3.8}$$

$$u^{//}(\xi) = -\ \lambda\,\mu^2\, m^2\ \ cos^{\,m}\,(\mu\,\xi\,) +\ \lambda\,\mu^2\, m(m-1)\, cos^{\,m-2}\,(\mu\,\xi\,). \quad (3.9)$$

and for (3.6) we use

$$u(\xi) =\ \lambda\ \ sin^{\,m}\,(\mu\,\xi\,), \qquad (3.10)$$

to get

$$u^{/}(\xi) =\ \ \lambda\,\mu\, m\ \ sin^{\,m-1}\,(\mu\,\xi\,)\cos\,(\mu\xi\,), \qquad (3.11)$$

$$u^{//}(\xi) = -\ \lambda\,\mu^2\, m^2\ \ sin^{\,m}\,(\mu\,\xi\,) +\ \lambda\,\mu^2\, m(m-1)\, sin^{\,m-2}\,(\mu\,\xi\,). (3.12)$$

and so on for other derivatives.

Substituting (3.7) – (3.9) or (3.10) – (3.12) into the reduced ODE gives an algebraic equation of $cos^{\,k}\,(\mu\,\xi\,)$ or $sin^{\,k}\,(\mu\,\xi\,)$ terms.

The parameters are then calculated by first balancing the exponents of each pair of cosine or sine terms and collecting all terms with the same power and setting their coefficients to zero to obtain a system of algebraic equations among the unknown. The problem is now completely reduced to a system of algebraic equations that can be easily solved to determine the solutions proposed in (3.5) and in (3.6) [87-91].

### 3.2.4 The AKNS System and BTs for NLEEs

In this approach, the Bäcklund transformations (BTs) for nonlinear evolution equations (NLEEs) are constructed through Ablowitz et. al. and Ricatti's form of the inverse method, as the following steps show [38-48]:

Consider the following AKNS eigenvalues problem:

$$\begin{aligned} \Phi_X &= P\Phi, \\ \Phi_t &= Q\Phi, \end{aligned} \tag{3.13}$$

where $\Phi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$, $P$ and $Q$ are $2 \times 2$ null-trace matrices

$$P = \begin{bmatrix} \eta & q \\ r & -\eta \end{bmatrix}, \quad Q = \begin{bmatrix} A & B \\ C & -A \end{bmatrix} \tag{3.14}$$

and $\eta$ is a parameter independent of $x$ and $t$, while $q$ and $r$ are assumed to be functions of $x$ and $t$. $A, B$ and $C$ are assumed to be functions of $\eta, q$ and $r$.

From Eqs. (2.13) and (2.14 we get the subsequent scattering problem:

$$\frac{\partial \varphi_1}{\partial x} - \eta\varphi_1 = q(x,t)\varphi_2 \tag{3.15}$$

$$\frac{\partial \varphi_2}{\partial x} + \eta\varphi_2 = r(x,t)\ \varphi_1 \tag{3.16}$$

in which the functions $\varphi_1$ and $\varphi_2$ evolve in time according to

$$\frac{\partial \varphi_1}{\partial t} = A(x,t;\eta)\varphi_1 + B(x,t;\eta)\varphi_2 , \tag{3.17}$$

$$\frac{\partial \varphi_2}{\partial t} = C(x,t;\eta)\varphi_1 - A(x,t;\eta)\varphi_2 , \tag{3.18}$$

The integrability condition reads

$$P_t - Q_x + PQ - QP = 0 \tag{3.19}$$

or in component form

$$\begin{aligned} &-A_x + qC - rB = 0 \\ &q_t - B_x - 2Aq + 2\eta B = 0 \\ &r_t - C_x - 2\eta C + 2Ar = 0 \end{aligned} \tag{3.20}$$

Consider the function

$$\Gamma = \frac{\varphi_1}{\varphi_2} \tag{3.21}$$

Differentiating Eqs. (3.21) with respect to $x$ and $t$, respectively , and using Eqs. (3.15), (3.16), (3.20), reduces Eqs. (3.13) to the Riccati equations:

$$\frac{\partial \Gamma}{\partial x} = 2\eta\Gamma + q - r\Gamma^2,$$

$$\frac{\partial \Gamma}{\partial t} = B + 2A\Gamma - C\Gamma^2 \tag{3.22}$$

We construct a transformation $\Gamma'$ which satisfies the same equation as (3.22) with potential $u'(x)$ where

$$u'(x) = u(x) + F(\Gamma, \eta), \tag{3.23}$$

is a new solution of the corresponding NLEE. In chapter 6, the AKNS system and new forms for $A, B, C, q$ and $r$ for some NLEEs are illustrated.

### 3.3 Ion-acoustic waves

An ion acoustic wave is a form of longitudinal oscillation in a plasma which is identical to an acoustic wave of neutral gas. Since the waves propagate via positively charged ions, acoustic ion waves can interfere with their electromagnetic field.

The ion-acoustic wave can occur in plasmas that are both collision-free and collisional. Physically, these waves are driven by electron pressure and ion inertia in a collisionless and non-isothermal plasma where the electron temperature is much higher than the ion temperature (Te ≫ Ti), while the coupling between the species is achieved by the electrostatic forces. While the dispersion relationship remains close to the collisionless situation, the physics of ion-acoustic waves in a collision-dominated plasma is more complicated as both electrostatic and collision-dominated effects come into play [92-106].

### 3.3.1 Plasmas

A Plasma is a state of matter in which many of the electrons freely wander between atom nuclei. The term plasma in physics designates a totally or partially ionised gas made up of electrons and ions. Through a phenomenological viewpoint, the definition of plasma as a new state of matter can be explained as the splitting of neutral atoms into electrons and ions at high temperatures is correlated with a new barrier of force, the ionisation force. The past 20 years have shown that plasma systems can reach liquid, gaseous and even solid phases.

As an electrically conductive medium, the plasma state possesses a number of new properties which distinguish it from neutral gases and liquids. Here in a solar prominence one can think of the ragged form of a lightning discharge or the magnetically confined plasma. The bulk of visible matter in space is in the state of plasma. This is definitely true as referring to the mass of planets, as the mass of stars. Dark matter will take the lead as opposed to plasmas if dark matter is exits.

A significant number of electrons in plasma have such high levels of energy that they cannot be retained by any nucleus. The atom that has lost some of its electrons and thus gained an electrical charge is called an ion. It is a particular type of ionized gas and is generally made up of:

- positively charged ions (‘positive ions’),
- electrons, and
- neutrals (atoms, molecules, radicals).

The plasma also may contain negative ions under special conditions, but this case will not be addressed. Positive ions may be packed or loaded individually. The ion population is sufficient to describe the ion density $n_i$ for a plasma, which contains only individually charged ions,

$$n_i = \frac{number\ of\ particles}{volume}, \qquad [n_i] = cm^{-3}\ or \quad [n_i] = m^{-3}\,. \ (3.24)$$

Besides the ion density, electron density $n_e$ and the neutral density $n_a$ characterize the plasma.

The following are the array of dynamic ion-acoustic wave equations with the quantum hydrodynamic model. The continuity and momentum equations for the ion fluid are given by [111-116]:

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} ( n_i \, u_i ) = 0, \tag{3.25}$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = - \frac{e}{m_i} \frac{\partial \emptyset}{\partial x}. \tag{3.26}$$

The dynamic equation for the inertialess electron quantum fluid is described by

$$e \frac{\partial \emptyset}{\partial x} - \frac{1}{n_e} \frac{\partial p_e}{\partial x} + \frac{h^2}{2 \, m_e} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{n_e}} \frac{\partial^2}{\partial x^2} \sqrt{n_e} \right) = 0. \tag{3.27}$$

The Poisson equation is written as

$$\frac{\partial^2 \emptyset}{\partial x^2} = \frac{e}{\epsilon_0} ( n_e - n_i ) = 0, \tag{3.28}$$

where $\emptyset$ is the electrostatic potential.

The density and velocity of the ion fluid are represented by $n_i$ and $u_i$ respectively, while $n_e$ is the electron fluid density. Also, $m_e$ and $m_i$ are the electron and ion masses, $-e$ is the electronic charge, $\in_0$ and $h$ are the dielectric and scaled Planck's constants. In equilibrium, we have $n_{e0} = n_{i0} = n_0$. Here $p_e$ is the electron pressure and $p_e\ (n_e\ )$ is found from the equation of state for the electron fluid [128-136].

## 3.4 Magnetohydrodynamics

The most commonly used equations for the analysis of inhomogeneous plasmas are the fluid equations. Magnetohydrodynamics (MHD) is the science of electromagnetic field interaction and the movement of molten semiconductors, liquid metals, and plasmas.

### 3.4.1 Governing equations for ideal MHD flows

Set of equations, written in standard notation and SI units [100-102; 109; 110], control the unstable-state ideal MHD flows:

The momentum equation

$$\rho\left(\frac{\partial\, \boldsymbol{v}}{\partial\, t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}\right) = -\,\boldsymbol{\nabla}\,p + \boldsymbol{J}\,\times\,\boldsymbol{B}, \tag{3.29}$$

the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{3.30}$$

Faraday's law

$$\nabla \times \boldsymbol{E} = -\frac{\boldsymbol{\partial B}}{\boldsymbol{\partial t}}, \tag{3.31}$$

Ampére's law

$$\nabla \times \boldsymbol{B} = \boldsymbol{\mu} \ \boldsymbol{J}, \tag{3.32}$$

the divergence-free Gauss law

$$\nabla \cdot \boldsymbol{B} = \mathbf{0}, \tag{3.33}$$

and Ohm's law

$$\mathbf{E} + \boldsymbol{v} \times \boldsymbol{B} = \mathbf{0}, \tag{3.34}$$

The symbols $\rho$, $\boldsymbol{v}$, $p$, $\boldsymbol{J}$, $\boldsymbol{B}$, $\boldsymbol{E}$ and $\mu$ denote the mass density, velocity field, pressure, electric current density, magnetic field, electric field and permeability constant, respectively. While there is no such thing as an incompressible fluid, we are using this concept whenever the shift in pressure density is so low that it is negligible.

In the incompressible case $\frac{d\rho}{dt} = 0$, therefore, using the mass conservation Eq. (3.30) yields. $\frac{d\rho}{dt} + \rho \, \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, so that

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \mathbf{0}. \tag{3.35}$$

### 3.4.2 Steady-state (Stationary) flows

The steady-state ideal MHD equilibrium of plasma flows is governed by the following set of equations [27]:

$$\rho \, (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \, \boldsymbol{v} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B}, \tag{3.36}$$

$$\boldsymbol{\nabla} \cdot (\rho \, \boldsymbol{v}) = 0, \tag{3.37}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0, \tag{3.38}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu \, \boldsymbol{J}, \tag{3.39}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{3.40}$$

$$\mathbf{E} + \boldsymbol{v} \times \boldsymbol{B} = 0. \tag{3.41}$$

The quantities $\rho$, $\mu$, $p$, $\boldsymbol{E}$, $\boldsymbol{v}$, $\boldsymbol{J}$ and $\boldsymbol{B}$ are the mass charge density, magnetic permeability, pressure, the electric field, velocity of the fluid, the current density, and the magnetic field respectively.

### 3.5 Isothermal magnetostatic atmospheres

The magnetostatic equations were widely used to model the magnetic structure of the sun [27−31]. Construction of isothermal magnetostatic models of the solar atmosphere is of general interest. The magnetostatic equations describe magnetised plasma in which mechanical equilibrium consists of pressure, magnetic, and external forces.

In solar physics, magnetostatic equations are used to model various phenomena, such as the slow evolution of solar flares, prominence formation, and magnetostatic support (the original model for the support of a plasma in a curved magnetic field against gravity) [32, 33]. In several cases [34 − 36], the problem of nonlinear equilibrium has been solved. Nonlinear evolution equations play a significant role in mathematical physics in various fields such as geochemistry, plasma physics, optical fibres, chemical kinematics, solid-state physics, chemical physics, and fluid mechanics. Wave dispersion, reaction, dissipation, diffusion, and convection phenomena are very common in the study of nonlinear waves [61-67].

### 3.5.1 Magnetostatics

Magnetostatics is the study of magnetic fields in systems with steady (not changing with time) currents. The magnetostatic field is a product of electrical current densities the way charging densities product just as the electrostatic field is.

More precisely, when the origin of such fields are known a magnetostatic model aims to test magnetic fields. The two potential magnetic field sources are (i) coils that bear electrical currents, and (ii) permanent magnets.

Maxwell's equations deduce the fundamental relations of magnetostatic by removing derivatives with respect to time. In these cases the electricity and magnetism equations are decoupled, leading to separate electrostatic and magnetostatic studies [59-61].

### 3.5.2 Basic equations

The equations governing magnetohydrostatic equilibrium [61-63] consist of

$$\mathbf{J} \times \boldsymbol{B} \;=\; \rho\, \boldsymbol{\nabla}\, \psi + \boldsymbol{\nabla}\, P, \tag{3.42}$$

which is coupled with Maxwells equations:

$$\boldsymbol{J} = \frac{\boldsymbol{\nabla} \times \boldsymbol{B}}{\mu}, \tag{3.43}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3.44}$$

where $\rho$, $\psi$, $\mu$ and $\psi$ are the mass density, the gravitational potential, the magnetic permeability and the gas pressure, respectively. It is assumed that the temperature is uniform in space and that the plasma is an ideal gas with state equation $P = \rho R_0 T_0$, where $R_0$ is the gas constant and the temperature is $T_0$.

# Chapter 4
# Soliton solutions for ion acoustic waves in plasmas

## 4.1 Introduction

A Plasma is a quasi-neutral gas of charged and neutral particles where interatomic and intermolecular forces are regulated by long-range electromagnetic interactions. About 99 per cent of the universe's known matter is in the plasma state.

Even the earth’s atmosphere slowly transitions from an uncharged troposphere to a highly charged ionosphere at a height of about 80 km above the surface. This region is characterised by significant numbers of charged atoms and molecules, and free electrons.

With a further increase in height, comes a further increase in the proportion of ionized matter. The magnetosphere and interplanetary space of the Earth are dominated by ionized particles that strongly interact with the magnetic field of the earth.

In astrophysical, spatial and laboratory settings, plasmas are omnipresent [17-22]. The nearest natural regions dominated by plasmas is Earth’s ionosphere and magnetosphere. Some characteristics such as density and temperature of different plasma regions in our solar system widely differ from each other. The physics of plasma sheaths is an intriguing subject in fundamental plasma physics and has been widely studied.

Classical plasma physics has focused primarily on high temperature and low-density environments, where no role is played by quantum mechanical effects. Nevertheless, recent advances in technology, such as the miniaturization of semiconductor devices and nanoscale artefacts, have made it possible to preview the application of plasma physics where the quantum effects may show up. Also, research on experiments under microgravity and optical trapping provide special conditions for plasmas to show complex behavior.

The attention of plasma physicists has been captured by dusty and complex plasmas, especially in what concerns technological applications. In this study, we present the main features of quantum collision-free plasma modulation and an original attempt to modulate certain special oscillations in two-dimensional dusty structures, recently reported in experimental and theoretical review papers.

Quantum plasma description is made through Wigner-Moyal formalism and equivalently through Schrödinger-like equations, both of which can be simplified to fluid equations [23-26]. The study of the oscillations in complex plasmas is suggested by defining a plasma form factor for the single Plasmon modes.

We will be addressing ionic acoustic waves in this section, which are the equivalent of sound waves in plasmas. Such waves spread because in a tenuous plasma, the more mobile electrons protect the ion electric field and set up their own field to move the ions. Such waves may be excited, but there is infrequent collision between the electrons and ions. For these waves, the external magnetic field is zero.

The equations of KdV and mKdV are derived and analytically analysed. The basic features of the soliton solutions from the equations will then be analysed. It was found that the under consideration degenerate plasma system facilitates the propagation of the solitons obtained from KdV and mKdV equations solutions.

In Sec. 4.2 of this chapter we will develop the appropriate plasma model. In Section 4.3, we will derive the differential equations describing ion-acoustic waves in plasmas and derive the KdV formula from a simplified PDE system describing ion-acoustic waves in plasma and effective solutions. In Section 4.4, we derive the KdV formula from a simplified model describing ion acoustic waves in a plasma. Precise solutions will be generated and analysed in Section 4.5.

## 4.2 Plasma Model

A plasma is an ionised fluid consists of positively charged ions and negatively charged electrons that interact and create electro-magnets. Plasmas produce waves similar to sound waves in pure compressible water, but these waves are scattered by the action of ions and electron oscillations.

Here we look at a two-fluid plasma model that considers as separate fluids ions and electrons. The equations describing the plasma couples the fluid ion and electron motion equations with Maxwell's electro-magnetic field equations.

We must find relatively low frequency waves that include ion movement, and we presume that magnetic fields are not present. Let $n^i$ , $n^e$ denote the ion and electron particle numbers respectively, $u^i, u^e$ their velocities, $p^i, p^e$ their pressures, and $E$ the electrical field. The mass and momentum conservation equations of the ion fluid for one spatial dimension [31] are given by

$$n_t^i + \left(n^i u^i\right)_x = 0, \tag{4.1}$$

$$m^i n^i \ (u_t^i + u^i u_x^i) + p_x^i = e\, n^i\, E. \tag{4.2}$$

Where $m^i$ is the ion's mass and $e$ is ion's charge. We consider this for simplicity to be the same as charging an electron. It is understood that the ion-fluid is 'cold' which means we are neglecting its pressure. Setting $p^i = 0$, we get [34-37]

$$n_t^i + \left(n^i u^i\right)_x = 0, \tag{4.3}$$

$$m^i \ (u_t^i + u^i u_x^i) = e\ E. \tag{4.4}$$

The equations of conservation of mass and momentum for the electron fluid are

$$n_t^e + (n^e u^e)_x = 0, \tag{4.5}$$

$$m^e n^e \ (u_t^e + u^e u_x^e) + p_x^e = -\, e\, n^e\, E, \tag{4.6}$$

Where $m^e$ is the electron's mass and $-e$ is charge of an electron. The electrons are also much lighter than the ions, so that their inertia can be ignored. Setting $m^e = 0$, we get

$$p_x^e = -\, e\, n^e\, E. \tag{4.7}$$

This calculation offers the electron density equation $n^e$. Then the electron velocity $u^e$ is determined from the mass conservation equation and is uncoupled from the remaining variables so we will not have to consider it any further.

For the electron fluid, we assume an isothermal state equation, meaning

$$p^e = k\, T\, n^e, \tag{4.8}$$

where $k$ is a constant of Boltzmann and temperature is $T$. Using (4.8) in (4.7) and writing $E = -\emptyset_x$ in term of an electrostatic potential $\emptyset$, we get

$$k\, T\, n_x^e = e\, n^e \emptyset_x\, . \tag{4.9}$$

This equation is implying that $n^e$ is given in term of $\emptyset$ by

$$n^e = n_0 \exp\left(\frac{e\ \ \emptyset}{k\ \ T}\right), \tag{4.10}$$

where the constant $n_0$ is the electron number density at $\emptyset = 0$.

Maxwell's equation generated by a charge density $\sigma$ is $\in_0 \quad \nabla \cdot \boldsymbol{E} = \sigma$ where $\in_0$ for the electrostatic field $\boldsymbol{E}$ is a dielectric constant. This equation does imply that

$$\in_0 \quad E_x = e\left(n^i - n^e\right). \tag{4.11}$$

In term of the potential ∅, equation (4.11) will be

$$- \ \emptyset_{xx} = \frac{e}{\in_0}\left(n^i - n^e\right), \tag{4.12}$$

Then, we may use (4.10) to eliminate $n^e$ from (4.12). Dropping the $i$-superscript on the ion-variables $(n^i, u^i)$, we can write the last equations for $(n, u, \emptyset)$ as

$$n_t + (n\,u)_x = 0, \tag{4.13}$$

$$u_t + u\,u_x + \frac{e}{m}\emptyset_x = 0, \tag{4.14}$$

$$- \ \emptyset_{xx} + \frac{e\,n_0}{\in_0}\exp\left(\frac{e \ \ \emptyset}{k \ \ T}\right) = \frac{e}{\in_0}\,n. \tag{4.15}$$

This equation contains a pair of $(n, u)$ evolution equations coupled with a semi-linear elliptic equation for ∅. In order to nondimensionalize these equations, we introduce the Debye length $\lambda_0$, and the ion-acoustic speed of sound $c_0$, defined by

$$\lambda_0^2 = \frac{\epsilon_0 \; k\,T}{n_0 e^2}, \qquad c_0^2 = \frac{k\,T}{m}. \tag{4.16}$$

In different conditions, these parameters vary by order of magnitudes for plasmas. Such as, in a dense plasma in laboratory, might we have $n_0 \approx 10^{20} m^{-3}$, $T \approx 60,000K$ and $\lambda_0 \approx 10^{-6}\; m$ ; in the wind of solar around the earth, we have $n_0 \approx 10^7 m^{-3}$, $\lambda_0 \approx 10\; m$ and $T \approx 120,000K$. Introducing the dimensionless variables

$$\bar{x} = \lambda_0^{-1}\, x\,, \qquad \overline{t} = \lambda_0^{-1} c_0\, t, \;\; \bar{n} = \frac{n}{n_0}, \;\; \bar{u} = \frac{u}{n_0},$$

$$\overline{\emptyset} = \frac{e\,\emptyset}{k\;\,T}, \qquad \rho = \exp(\emptyset) \tag{4.17}$$

and drop the bars, we are getting the nondimensionalized equations

$$n_t + (n\,u)_x = 0, \tag{4.18}$$

$$u_t + u\,u_x + \;\emptyset_x = 0, \tag{4.19}$$

$$\emptyset_{xx} + \; n = \; \rho. \tag{4.20}$$

## 4.3 Derivation of the KdV equation and Soliton solutions

In this section, we utilize the KdV equation to generate exact solutions for the simplified system (4.18) -(4.20) describing ion-acoustic waves in a plasma. This derivation illustrates the universal nature of the KdV equation, which applies to any wave motion with weak advective nonlinearity and weak long wave dispersion.

To study the properties of soliton ion-acoustic waves in a plasma, we apply the reductive perturbation technique to the basic equations following Washimi and Taniuti [119].

Consequently, space and time coordinates are stretched over the following relations,

$$\xi = \sqrt{\epsilon}(x - \lambda\, t), \qquad \tau = \sqrt{\epsilon^3}\; t, \tag{4.21}$$

where $\in$ is a small positive parameter and $\lambda$ is the wave velocity. By using (4.21), equations (4.18)-(4.20) become

$$-\lambda n_\xi + (n\;\; u)_\xi + \in\; n_\tau = 0, \tag{4.22}$$

$$-\lambda\, u_\xi + \;\; \emptyset_\xi + u\, u_\xi + \in\; u_\tau = 0, \tag{4.23}$$

$$\rho - \in \emptyset_{\xi\xi} = n. \tag{4.24}$$

We look for asymptotic solutions for equations (4.22) – (4.24) of the form

$$n = 1 + \in n_1 + \in^2 n_2 + \cdots, \tag{4.25}$$

$$u = \in u_1 + \in^2 u_2 + \cdots, \tag{4.26}$$

$$\emptyset = \in \emptyset_1 + \in^2 \emptyset_2 + \cdots, \tag{4.27}$$

$$\rho = e^{\emptyset} = 1 + \in \emptyset_1 + \in^2 \left(\emptyset_2 + \frac{\emptyset_1^2}{2}\right) + \cdots. \tag{4.28}$$

Using these expansions in (4.25) – (4.28), we obtain

$$(u_{1\xi} - \lambda n_{1\xi}) \in + \left(u_{2\xi} - \lambda n_{2\xi} + (n_1 u_1)_\xi + n_{1\tau}\right) \in^2 + \cdots = 0,$$

$$(-\lambda\, u_{1\xi} + \emptyset_{1\xi}) \in + \left(-\lambda\, u_{2\xi} + \emptyset_{2\xi} + u_1 u_{1\xi} + u_{1\tau}\right) \in^2 + \cdots = 0,$$

$$(\emptyset_1 - n_1) \in + (\emptyset_2 - n_2 + \frac{1}{2}\emptyset_1^2 - \emptyset_{1\xi\xi}) \in^2 + \cdots = 0.$$

Equating coefficients of $\in$ to zero, we find that

$$\begin{cases} u_{1\xi} - \lambda n_{1\xi} = 0, \\ -\lambda\, u_{1\xi} + \emptyset_{1\xi} = 0, \\ \emptyset_1 - n_1 = 0. \end{cases} \tag{4.29}$$

Equating coefficients of $\in^2$ to zero, we find that

$$
\begin{cases} u_{2\xi} - \lambda n_{2\xi} + (n_1 u_1)_\xi \ + \ n_{1\tau} = 0, \\ -\lambda\, u_{2\xi} + \ \emptyset_{2\xi} + u_1 u_{1\xi} + \ u_{1\tau} = 0, \\ \emptyset_2 \ - \ n_2 + \dfrac{1}{2}\emptyset_1^2 - \emptyset_{1\xi\xi} = \ 0. \end{cases} \tag{4.30}
$$

Eliminating $\emptyset_1$ from (4.29), we obtain

$$
\begin{cases} u_{1\xi} - \lambda n_{1\xi} = 0, \\ -\lambda\, u_{1\xi} + \ n_{1\xi} = 0. \end{cases} \tag{4.31}
$$

From (4.31), we get a homogeneous linear system for $(n_1, u_1)$,

$$
\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} n_1 \\ u_1 \end{pmatrix}_\xi = 0. \tag{4.32}
$$

This system has a nontrivial solution if $\lambda^2 = 1$. We assume that $\lambda = 1$ for definiteness, corresponding to a right-moving wave. Then

$$
\begin{pmatrix} n_1 \\ u_1 \end{pmatrix} = \ \psi(\xi, \tau) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \emptyset_1 = \ \psi(\xi, \tau), \tag{4.33}
$$

where $\psi(\xi, \tau)$ is an arbitrary scalar-valued function. At the next order, after setting $\lambda = 1$ and eliminating $\emptyset_2$ in (4.30), we obtain

$$
\begin{cases} u_{2\xi} - \lambda n_{2\xi} + (n_1 u_1)_\xi \ + \ n_{1\tau} = 0, \\ -\lambda\, u_{2\xi} + \ n_{2\xi} \ - \ \emptyset_1 \emptyset_{1\xi} + \emptyset_{1\xi\xi\xi} + u_1 u_{1\xi} + \ u_{1\tau} \ = 0. \end{cases} \tag{4.34}
$$

From (4.34), we obtain a nonhomogeneous linear system for $(n_2, u_2)$,

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} n_2 \\ u_2 \end{pmatrix}_\xi + \begin{pmatrix} (n_1 u_1)_\xi \ + \ n_{1\tau} \\ -\ \emptyset_1 \emptyset_{1\xi} + \emptyset_{1\xi\xi\xi} + u_1 u_{1\xi} + \ u_{1\tau} \end{pmatrix} = 0. \quad (4.35)$$

This system is solvable for $(n_2, u_2)$ if and only if the nonhomogeneous term is orthogonal to the null-vector $(1, 1)$. Using (4.35), we find that this condition implies that $\psi(\xi, \tau)$ satisfies a KdV equation

$$\psi_\tau + \psi\, \psi_\xi + \frac{1}{2} \psi_{\xi\xi\xi} = 0. \qquad (4.36)$$

Note that the linearized dispersion relation of this equation agrees with the long wave expansion of the linearized dispersion relation of the original system. If (4.36) satisfied, then we may solve (4.35) for $(n_2, u_2)$.

The solution is the sum of a solution of the nonhomogeneous equations and an arbitrary multiple $\psi_2(\xi, \tau) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ of the solution of the homogeneous problem. We may compute higher-order terms in the asymptotic solution in a similar way. At the order $\in^k$, we obtain a nonhomogeneous linear equation for $(n_k, u_k)$ of the form

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} n_k \\ u_k \end{pmatrix}_\xi + \begin{pmatrix} f_{k-1} \\ g_{k-1} \end{pmatrix} = 0, \qquad (4.37)$$

where $f_{k-1}$, $g_{k-1}$ depend only on $(n_1, u_1)$, $\cdots, (n_{k-1}, u_{k-1})$, and $\emptyset_k$ may be expressed explicitly in terms of $n_1, \ldots, n_k$ .

The condition that this equation is solvable for $(n_k, u_k)$ is $f_{k-1} + g_{k-1} = 0$ and this condition is satisfied if $\psi_{k-1}$ satisfies a suitable equation. The solution for$(n_k, u_k)$ then involves an arbitrary function of integration $\psi_k$ . An equation for $\psi_k$ follows from the solvability condition for the order $(k + 1)$ equations.

We find a travelling wave solution for the KdV equation (36) by using the sine-cosine method. We first use the wave variable $\theta = \xi - b\,\tau$ where $b$ is a constant, to carry a PDE in two independent variables (36) into the following ordinary differential equation

$$-b\,\psi^{/} + \psi\psi^{/} + \frac{1}{2}\psi^{///} = 0, \tag{4.38}$$

where $^{/} = \frac{d}{d\theta}$. Integrating (3.38) once, and considering the constants of integration as zero, we can find

$$-b\,\psi + \left(\frac{1}{2}\psi^{2}\right) + \frac{1}{2}\psi^{//} = 0, \tag{4.39}$$

We then take [88-91]

$$\psi(\theta) = a \ cos^{\,m}(\mu \, \theta \,), \tag{4.40}$$

$$\psi^{/}(\theta) = - \ a \, \mu \, m \ \ cos^{\,m-1}(\mu \, \theta \,) \sin(\mu \, \theta \,), \tag{4.41}$$

$$\psi^{//}(\theta) = - \ a \, \mu^2 m^2 cos^{\,m}(\mu \, \theta \,) + \ a \, \mu^2 \, m(m-1) cos^{\,m-2}(\mu \, \theta \,). \tag{4.42}$$

Substituting (4.40) – (4.42) into (4.39) yields

$$- \left( a \, b \ + \frac{1}{2} \, a \, \mu^2 m^2 \right) cos^{\,m}(\mu \, \theta \,) + \frac{1}{2} a^2 cos^{\,2m}(\mu \, \theta \,)$$

$$+ \frac{1}{2} \, a \, \mu^2 \, m(m-1) cos^{\,m-2}(\mu \, \theta \,) = 0. \tag{4.43}$$

Equating the exponents and the coefficients of each pair of the cosine functions, we find the following system of algebraic equations:

$$m \neq 0,$$

$$2m = m - 2,$$

$$a \, b \ + \frac{1}{2} \ a \, \mu^2 m^2 \ = 0, \tag{4.44}$$

$$\frac{1}{2} a^2 \ + \frac{1}{2} \, a \, \mu^2 \, m(m-1) = 0.$$

From system (4.44), we obtain

$$m = -2, \qquad \mu = \sqrt{\frac{-b}{2}}, \qquad a = 3b\,. \tag{4.45}$$

Then, we obtain the formal solitary wave solutions for the equation (4.36)

$$\psi(\xi,\tau) = \ 3b \ \ sech^2 \frac{1}{2}\left(\sqrt{2b}\ (\xi - b\ \tau)\ \right), \quad b > 0 \tag{4.46}$$

The steady state solution (4.46) of the KdV equation is obtained. The basic characteristics of the solitary waves as they evolve in time $t$ have been obtained analytically and graphically (see Figures 4.1 & 4.2).

It is found that the amplitude of the soliton does not depend on the external magnetic field but depends on the ion and electron temperature parameter. On the other hand, the width of the soliton depends on the strength of external magnetic field.

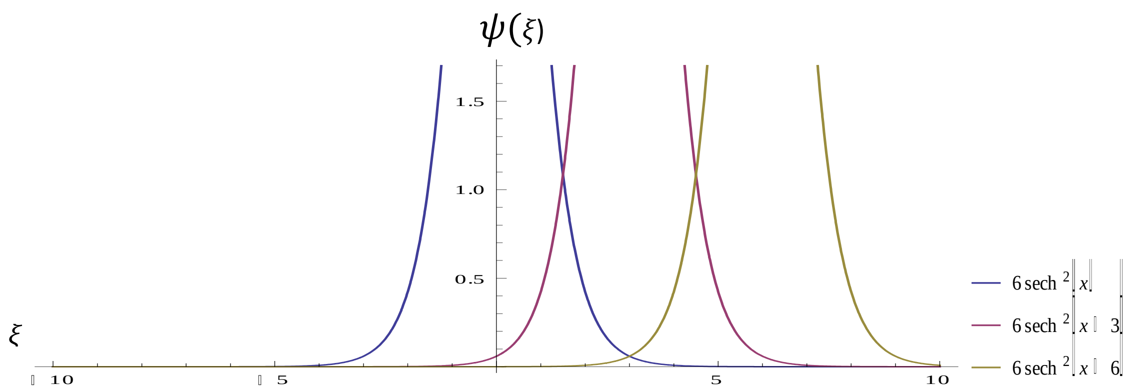



Figure 4.1: Travelling wave solutions of eq. (4.46) with $t = 0$ (blue), $\mathrm{t} = 1$ (red) and $\mathrm{t} = 2$ (oily)

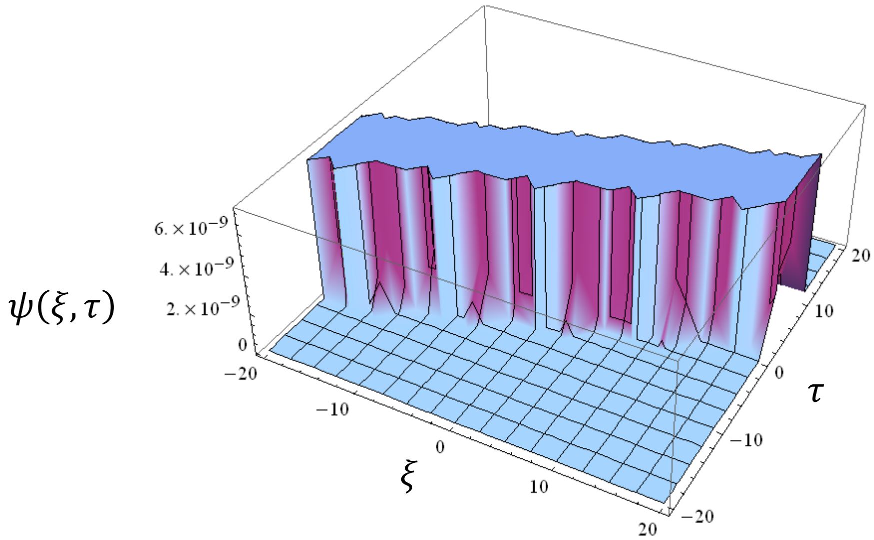



Figure 4.2: Bell − type wave solution of eq. (4.46) with $\lambda = 1$,

$\in = 1$ and $b = 2$

For b <0 we get

$$\psi(\xi,\tau) = 3b\, sec^2 \frac{1}{2}\left(\sqrt{-2b}\ (\xi - b\,\tau)\right), \quad b < 0. \tag{4.47}$$

Figures 4.3 & 4.4 show how the different parameters play a key role in modifying the behavior of solitary waves in magnetized plasma system. It is evident that higher magnitudes of $b$ cause significant reductions in the amplitude of the solitary wave, but the soliton width increases as the width of $b$ increases.

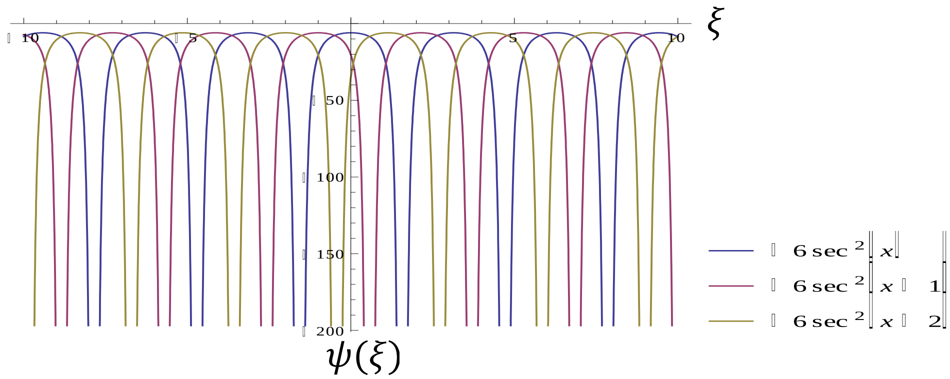



Figure 4.3: Solitary wave of bell types of eq. (4.47) with $t = 0$ (blue),

$\mathrm{t} = 1$ (red) and $\mathrm{t} = 2$(oily)

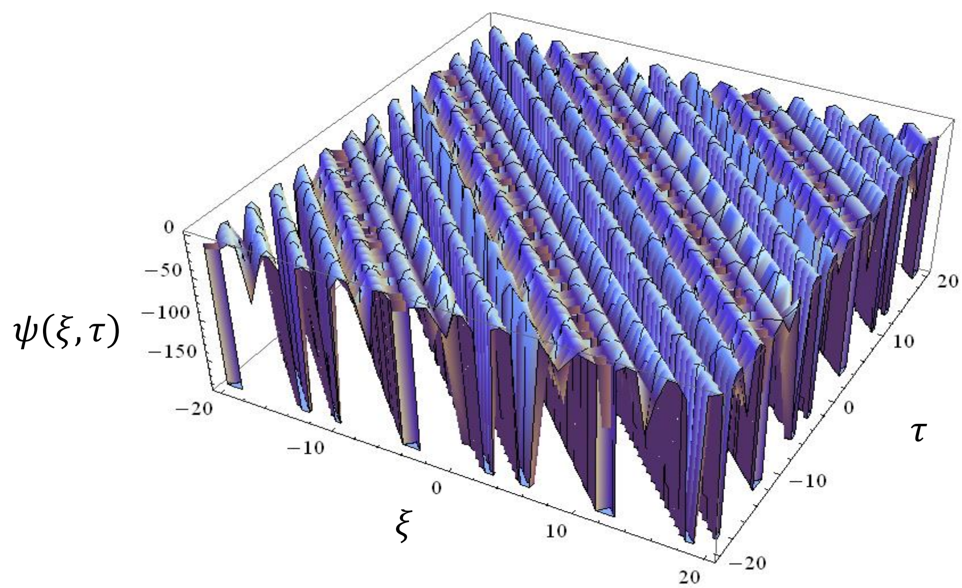



Figure 4.4: Dip and Hump soliton solution of eq. (4.47) with $\lambda = 1$,

$\in = 1$ and $\mathrm{b} = -2$

We can now write the solution of the system (4.18) – (4.20) as

$$\begin{pmatrix} n \\ u \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3b \in sech^2 \frac{1}{2}\left(\sqrt{2b}\,(\xi - b\,\tau)\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + O(\in^2) \quad (4.48)$$

if b>0, and

$$\begin{pmatrix} n \\ u \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \ 3b \ \in \ sec^2 \frac{1}{2}\left(\sqrt{-2b}(\xi - b\,\tau)\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + O(\in^2) \quad (4.49)$$

if b<0.

In nutshell, the leading-order asymptotic solution of (4.18) – (4.20) as $\in \to 0^+$ is

$$\begin{pmatrix} n \\ u \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \in \ \psi \ \left(\sqrt{\in}\ (x-t), \sqrt{\in^3}\ \ t\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + O(\in^2) \qquad (4.50)$$

We expect that this asymptotic solution is valid for long times of the order

$\tau = O(1)\ \ or \ \ t = O\left(\epsilon^{\frac{-3}{2}}\right)$

**4.4 Derivation of the mKdV equation and Soliton solutions**

In this section, we utilize the mKdV equation to generate solutions for the simplified system (3.18)-(3.20) describing ion-acoustic waves in plasmas. This derivation shows the universal nature of the mKdV formula, which applies to any wave movement with strong adventive nonlinearity and weak long wave dispersion.

Thus, space and time coordinates are stretched through the following relations,

$$\xi = \in \ (x - \lambda\, t), \qquad \tau = \in^3 \ t, \tag{4.51}$$

where $\in$ is a small positive parameter and $\lambda$ is the wave velocity. By using (4.51), equations (4.18)-(4.20) become

$$-\lambda n_\xi + (n\ u)_\xi + \in^2 n_\tau = 0, \tag{4.52}$$

$$-\lambda\, u_\xi + \ \emptyset_\xi + u\, u_\xi + \in^2 u_\tau = 0, \tag{4.53}$$

$$\rho \ - \in^2 \ \emptyset_{\xi\xi} - \ n = 0. \tag{4.54}$$

We look for asymptotic solutions for (4.52) – (4.54) of the form

$$n = 1 + \in n_1 + \in^2 n_2 + \in^3 n_3 + \cdots, \tag{4.55}$$

$$u = \in u_1 + \in^2 u_2 + \in^3 u_3 + \cdots, \tag{4.56}$$

$$\emptyset = \in \emptyset_1 + \in^2 \emptyset_2 + \in^3 \emptyset_3 + \cdots, \tag{4.57}$$

$$\rho = 1 + \in \emptyset_1 + \in^2 \left(\emptyset_2 + \frac{\emptyset_1^2}{2}\right) + \in^3 \left(\emptyset_3 + \emptyset_1 \emptyset_2 + \frac{\emptyset_1^3}{6}\right) + \cdots. \tag{4.58}$$

Using these expansions in (4.55) – (4.58), and equating coefficients of $\in$, we find that

$$u_{1\xi} - \lambda n_{1\xi} = 0, \tag{4.59}$$

$$-\lambda\, u_{1\xi} + \; \emptyset_{1\xi} = 0, \tag{4.60}$$

$$\emptyset_1 - n_1 = 0. \tag{4.61}$$

Equating coefficients of $\in^2$, we find that

$$u_{2\xi} - \lambda n_{2\xi} + (n_1 u_1)_\xi = 0, \tag{4.62}$$

$$-\lambda\, u_{2\xi} + \; \emptyset_{2\xi} + u_1 u_{1\xi} = 0, \tag{4.63}$$

$$\emptyset_2 - n_2 + \frac{1}{2}\emptyset_1^2 = 0. \tag{4.64}$$

Equating coefficients of $\in^3$, we find that

$$u_{3\xi} - \lambda n_{3\xi} + n_{1\tau} + (n_1 u_2)_\xi + (n_2 u_1)_\xi = 0, \tag{4.65}$$

$$-\lambda\, u_{3\xi} + \; \emptyset_{3\xi} + u_{1\tau} + (u_1 u_2)_\xi = 0, \tag{4.66}$$

$$\emptyset_3 - n_3 + \emptyset_1\emptyset_2 + \frac{1}{6}\emptyset_1^3 - \emptyset_{1\xi\xi} = 0. \tag{4.67}$$

Eliminating $\emptyset_1$ from (4.59) – (4.61), we get a homogeneous linear system for $(n_1, u_1)$,

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} n_1 \\ u_1 \end{pmatrix}_\xi = 0. \tag{4.68}$$

This system has a nontrivial solution if $\lambda^2 = 1$. We suppose that $\lambda = 1$ for definiteness, corresponding to a right-moving wave. Then

$$\begin{pmatrix} n_1 \\ u_1 \end{pmatrix} = g(\xi,\tau)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad u_1 = g(\xi,\tau), \tag{4.69}$$

where $g(\xi,\tau)$ is an arbitrary scalar-valued function. At the next order, after setting $\lambda = 1$ and eliminating $\emptyset_2$ in (4.62) – (4.64), we obtain a nonhomogeneous linear system for $(n_2, u_2)$,

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} n_2 \\ u_2 \end{pmatrix}_\xi + \begin{pmatrix} (n_1u_1)_\xi \\ -\emptyset_1\emptyset_{1\xi} + u_1u_{1\xi} \end{pmatrix} = 0. \tag{4.70}$$

Again we suppose that $\lambda = 1$ for definiteness, corresponding to a right-moving wave. Then

$$\emptyset_1 = \sqrt{3}\;\; g(\xi,\tau), \quad \emptyset_2 = -g^2(\xi,\tau), \quad n_2 = -u_2 = \frac{g^2(\xi,\tau)}{2}. \tag{4.71}$$

At the next order, after setting $\lambda = 1$ and eliminating $\emptyset_2$ in (4.65) – (4.67), we obtain a nonhomogeneous linear system for $(n_3, u_3)$,

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} n_3 \\ u_3 \end{pmatrix}_\xi + \begin{pmatrix} n_{1\tau} + (n_1u_2)_\xi + (n_2u_1)_\xi \\ \emptyset_{1\xi\xi\xi} - (\emptyset_1\emptyset_2)_\xi - \dfrac{\emptyset_1^2\emptyset_{1\xi}}{2} + u_{1\tau} + (u_1u_2)_\xi \end{pmatrix} = 0. \tag{4.72}$$

This system is solvable for $(n_3, u_3)$ if and only if the nonhomogeneous term is orthogonal to the null-vector $(1, 1)$. Using (4.72), we find that this condition implies that $g(\xi, \tau)$ satisfies a mKdV equation

$$g_\tau - \alpha\, g^2 g_\xi + \beta\, g_{\xi\xi\xi} = 0, \qquad \text{where} \qquad \alpha = \frac{3}{4}, \quad \beta = \frac{\sqrt{3}}{2}. \quad (4.73)$$

Note that the linearized dispersion relation of this equation agrees with the long wave expansion of the linearized dispersion relation of the original system.

If (3.73) satisfied, then we may solve (3.72) for$(n_3, u_3)$. The solution is the sum of a solution of the nonhomogeneous equations and an arbitrary multiple $g_3(\xi, \tau)\begin{pmatrix}1\\1\end{pmatrix}$ of the solution of the homogeneous problem. We may compute higher-order terms in the asymptotic solution in a similar way. At the order $\in^k$, we obtain a nonhomogeneous linear equation for $(n_k, u_k)$ of the form

$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}n_k\\ u_k\end{pmatrix}_\xi + \begin{pmatrix}f_{k-1}\\ g_{k-1}\end{pmatrix} = 0, \qquad (4.74)$$

where $f_{k-1},\ g_{k-1}$ depend only on $(n_1, u_1),\ \cdots, (n_{k-1}, u_{k-1})$, and $\emptyset_k$ may be expressed explicitly in terms of $n_1, \ldots, n_k$ . The condition that this equation is solvable for $(n_k, u_k)$ is $f_{k-1} + \ g_{k-1} = 0$, and this condition is satisfied if $g_{k-1}$ satisfies a suitable equation.

The solution for $(n_k, u_k)$ then involves an arbitrary function of integration $g_{\ k}$ . An equation for $g_k$ follows from the solvability condition for the order $(k \ + \ 1)$ equations. We can find a travelling wave solution for the mKdV equation (4.73) by using the sine-cosine method.

We first use the wave variable $\theta = \xi - b\,\tau$ where $b$ is a constant, to reduce the PDE in two independent variables (4.73) into the following ordinary differential equation

$$-\,b\ g^{/} - \alpha\ \ g^2 g^{/} + \ \ \beta\ g^{///} = 0, \tag{4.75}$$

where $^{/} \ = \frac{d}{d\theta}$ . Integrating (4.75) once, and considering the constants of integration as zero, we find that

$$-\,b\ g\ -\ \ \left(\frac{\alpha}{3} g^{\ 3}\right)\ +\ \ \beta\ g^{//} = 0, \tag{4.76}$$

We then take [88-91 ]

$$g(\theta) = \ a \ \ sin^{\ m}(\mu\ \theta\ ), \tag{4.77}$$

$$g^{/}(\theta) = a\ \ \mu\, m\ \ sin^{\ m-1}(\mu\ \theta\ )\cos\,(\mu\ \theta\ ), \tag{4.78}$$

$$g^{//}(\theta) = -\ a\,\mu^2 m^2 sin^{\,m}(\mu\,\theta\,) +\ a\,\mu^2\ m(m-1) sin^{\,m-2}(\mu\theta\,). \quad (4.79)$$

Substituting (3.77) – (3.79) into (3.76) yields

$$-\,(a\,b\ +\ \beta\ a\,\mu^2 m^2) sin^{\,m}(\mu\,\theta\,) - \frac{\alpha}{3} a^3 sin^{\ 3m}(\mu\,\theta\,)$$

$$+\beta\ a\,\mu^2\ m(m-1) sin^{\,m-2}(\mu\,\theta\,) = 0. \qquad (4.80)$$

Equating the exponents and the coefficients of each pair of the sine functions, we find the following system of algebraic equations:

$$m \neq 0,$$

$$3m = m-2,$$

$$a\,b\ +\ \beta\ a\,\mu^2 m^2\ = 0, \qquad (4.81)$$

$$\frac{-\,\alpha}{3} a^3\ + \beta\ a\,\mu^2\ m(m-1) = 0.$$

From this system (4.81), we obtain

$$m = -1, \qquad \mu = \sqrt{\frac{-b}{\beta}}, \qquad a\ = \sqrt{\frac{-\,6b}{\alpha}}\,. \qquad (4.82)$$

We can now write the formal solitary wave solutions for equation (4.73) as

$$g(\xi,\tau) = \sqrt{\frac{6b}{\alpha}}\,\mathrm{csch}\left(\sqrt{\frac{b}{\beta}}\,(\xi - b\,\tau)\right),\;\; b > 0 \qquad (4.83)$$

The mKdV equation admits solitary type solutions given by equation (4.83) which has a $\mathrm{csch}(\mu\,\theta\,)$ profile. The peak width and amplitude of these solitary waves are functions of several parameters. The presented Figures 4.5 & 4.6 may help us to understanding the study of nonlinear waves in astrophysical plasmas. Comparing these results with other wave profiles can add to our knowledge of plasma physics.

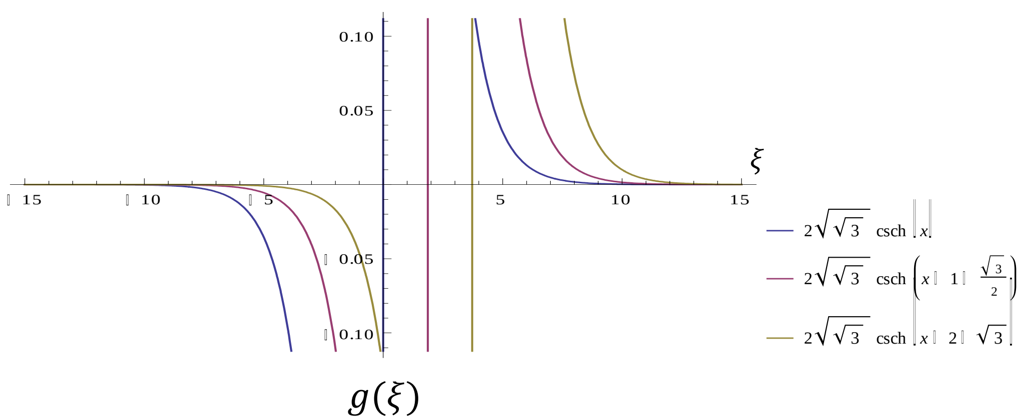



Figure 4.5: Travelling wave solutions of eq. (4.83) with $t = 0$ (blue), t = 1 (red) and t = 2 (oily)

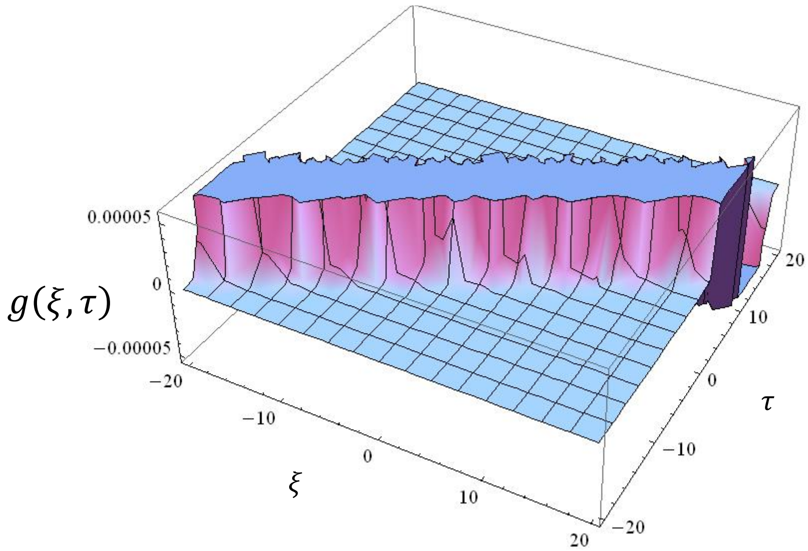



Figure 4.6: Hump soliton solution of eq. (4.83) with $\lambda = 1$,

$\in = 1$ and $b = \frac{\sqrt{3}}{2}$

In Figure 4.6, the motion of curves is intimately related to the mKdV equation. This leads to the existence of soliton-like solutions in the motion of curves, as well as the existence of infinite number of conservation laws that can be put into relation with global geometric quantities. The purpose of the figures is to describe these relations for the two dimensional and three-dimensional cases.

$$g(\xi,\tau) = \sqrt{\frac{-6b}{\alpha}}\csc\left(\sqrt{\frac{-b}{\beta}}\,(\xi - b\,\tau)\right), \quad b < 0. \tag{4.84}$$

Although the mKdV equation is incapable of describing shock waves, it can however describe the asymptotic behavior of non-steady, low-amplitude wave trains.

This means we have 2-soliton solutions of behavior and this is illustrated in Figures 4.7 & 4.8. We see that for the mKdV equation we can obtain solutions travelling at different relative speeds to each other and so we assume this is possible for quite a range of speeds, although we suspect that at some point the algebra might become too hard.

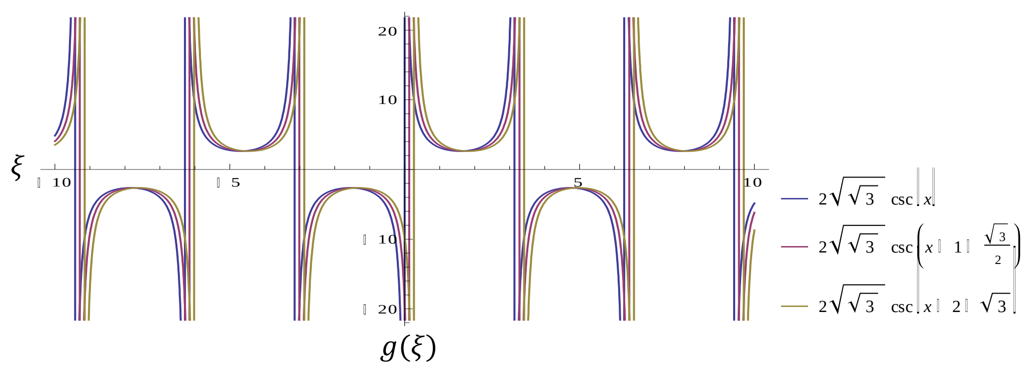



Figure 4.7: Solitary wave of bell and valley types of eq. (4.84) with $t = 0$ (blue), $t = 1$ (red) and $t = 2$(oily)

In Figure 4.7, we analyzed the solutions of the mKdV equation in various formulations used in theoretical and applied studies of equations with a similar structure. Some of the soliton solutions obtained were found to belong to well-known classes (breathers and kinks), while the others represented new forms for the given equation, namely, wobblers, pair breathers, and oscillatory waves.

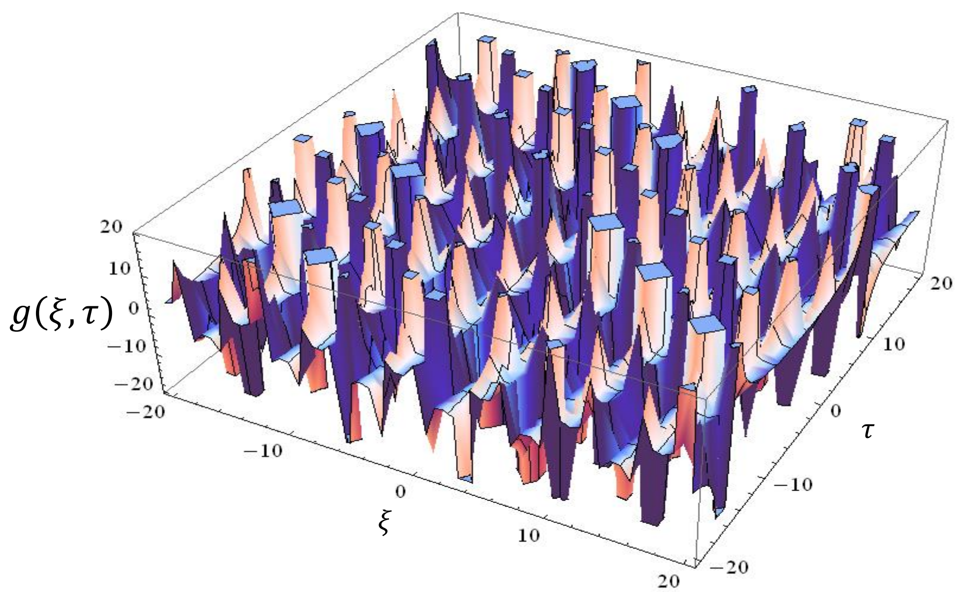



Figure 4.8: Dip and Hump soliton solution of eq. (4.84) with $\lambda = 1$, $\in = 1$ and $b = -\frac{\sqrt{3}}{2}$

In Figure 4.8, the soliton solutions covered all cases of double interactions and included triple interactions between breathers and kinks.

Hence, we can write the solution of the system (4.18) – (4.20) as

$$\begin{pmatrix} n \\ u \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sqrt{\frac{6b}{\alpha}} \in \operatorname{csch}\left(\sqrt{\frac{b}{\beta}}(\xi - b\,\tau)\right)\begin{pmatrix} 1 \\ 1 \\ \sqrt{3} \end{pmatrix} + O(\in^2) \qquad (4.85)$$

$$\begin{pmatrix} n \\ u \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sqrt{\frac{-6b}{\alpha}} \in \csc\left(\sqrt{\frac{-b}{\beta}}(\xi - b\,\tau)\right)\begin{pmatrix} 1 \\ 1 \\ \sqrt{3} \end{pmatrix} + O(\in^2) \qquad (4.86)$$

## 4.5 Summary

There has been considerable interest in new wave modes in plasmas recently and this is currently a rapidly growing area in plasma physics. The results indicate that the nonlinear evolution of charge fluctuations can have important effects on instabilities in plasmas that require further investigation. We have described a new simulation model that can be used for studying unique physical properties of waves in plasmas due to charging effects.

We also have studied the system of PDEs that describes ion acoustic waves in plasmas and obtained exact soliton solutions. The corresponding KdV and mKdV equations were obtained using the reductive perturbation method. The solutions may have important applications in plasmas in which the plasma ions drift with respect to the electrons. Ion acoustic waves solutions may also be important for structures observed in numerous space and laboratory plasmas.

# Chapter 5
# Travelling wave solutions for three-dimensional incompressible MHD

## 5.1 Introduction

The acronym MHD stands for three terms, magneto, indicating a magnetic field, hydro, referring to a liquid, and dynamics, meaning movement. The field of MHD is a fascinatingly rich field of physics and applied mathematics that considers the behavior of an electrically conducting fluid in the presence of an external electromagnetic field.

Although interesting in its own right, MHD also has numerous engineering and scientific applications. This range from the pursuit of reliable energy sources such as nuclear fusion [100], to understanding near-earth plasmas such as solar wind [101] and the more exotic astrophysical objects such as stars, black holes, and the interstellar medium [102].

Virtually in all of these areas, the phenomenon of turbulence and the role turbulence plays in engineering and scientific applications is certainly critical. As previously mentioned, MHD is concerned with the behavior of fluids in the presence of an external electromagnetic field. Of course, if the fluid does not conduct electricity, then it will not influence, nor will it be influenced by, the external electromagnetic field.

Since there is a vast amount of driving forces in MHD systems, there is a possibility for different types of waves that could propagate through the plasma. In incompressible MHD, we touch on the differences between ideal versus (visco-) resistive evolutions [110].

The Elsasser formulation of the governing equations allows insights to be generalized from linear to nonlinear wave package behavior, which prominently appear in MHD turbulence theories. High resolution (pseudo-) spectral simulations give important clues to the anisotropic nature of MHD turbulence.

We also discuss numerical evidence for a singular structure in incompressible MHD and for small-scale dynamo action by summarizing selected simulation-based studies. We end this chapter with an introduction to compressible MHD, where the linear wave picture is richer and allows for wave steepening, paving the way for shock-dominated plasma behavior [128].

The MHD description governs the large-scale dynamics of plasmas and applies to many laboratories as well as astrophysical configurations.

Incompressible MHD has traditionally focused on topics like MHD turbulence, dynamo aspects, and singular structure formation. We investigate the type of waves that can exist by linearizing the MHD equations and then applying Fourier transforms.

The field of incompressible MHD is a particularly rich subset of physics and applied mathematics. The challenges inherent in the equations provide a plethora of research opportunities. Aside from purely academic pursuits, MHD also plays an important role in the development of engineering technologies. Designing suitable engineering systems using electrically conducting fluids requires using computational techniques.

One of the most prominent reasons for this difficulty is the phenomenon of fluid turbulence which again rears its head in MHD [133,134]. In addition to the velocity field displaying disordered behavior, the external electromagnetic field quantities also display such behavior [126]. The main aim of this chapter is to use the travelling wave method in the construction of exact soliton solutions for three-dimensional incompressible MHD equations.

This chapter is organized as follows. In section 5.1, we review the main governing equations of incompressible MHD. In section 5.2, the $sin(k\xi) - cos(k\xi)$ method and the exact solutions for the incompressible MHD problem are presented. Finally, section 5.3 concludes the chapter.

## 5.2 Travelling wave solutions

We start with the equations of incompressible MHD,

$$\nabla \cdot \boldsymbol{V} = \nabla \cdot \boldsymbol{B} = 0, \tag{5.1}$$

$$\boldsymbol{V_t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} - (\boldsymbol{B} \cdot \nabla)\mathbf{B} + \nabla(P + \frac{1}{2}|\mathbf{B}|^2)$$

$$-v_1\boldsymbol{V}_{xx} - v_2\boldsymbol{V}_{yy} - v_3\boldsymbol{V}_{zz} = 0, \tag{5.2}$$

$$\boldsymbol{B}_t + (\boldsymbol{V} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{V}$$

$$-\eta_1\boldsymbol{B}_{xx} - \eta_2\boldsymbol{B}_{yy} - \eta_3\boldsymbol{B}_{zz} = 0\,, \tag{5.3}$$

where

$$\boldsymbol{V} = (V_1(x,y,z,t), V_2(x,y,z,t), V_3(x,y,z,t))^T$$

$$\boldsymbol{B} = (B_1(x,y,z,t), B_2(x,y,z,t), B_3(x,y,x,t))^T$$

$$\text{and} \qquad P = P(x,y,z,t)$$

represent the unknown velocity field, the magnetic field, and the pressure of the flow respectively, and $v_1, v_2, v_3, \eta_1, \eta_2, \text{and}\, \eta_3$ are the viscosity coefficients of the flow.

In scalar form, the three-dimensional incompressible MHD system (5.1)-(5.3) is

$$V_{1,x} + V_{2,y} + V_{3,z} = B_{1,x} + B_{2,y} + B_{3,z} = 0, \tag{5.4}$$

$$V_{1,t} + V_1 V_{1,x} + V_2 V_{1,y} + V_3 V_{1,z} + P_x + B_2(B_{2,x} - B_{1,y})$$

$$+B_3(B_{3,x} - B_{1,z}) - v_1 V_{1,xx} - v_2 V_{1,yy} - v_3 V_{1,zz} = 0, \tag{5.5}$$

$$V_{2,t} + V_1 V_{2,x} + V_2 V_{2,y} + V_3 V_{2,z} + P_y + B_1(B_{1,y} - B_{2,x})$$

$$+B_3(B_{3,y} - B_{2,z}) - V_1 V_{2,xx} - V_2 V_{2,yy} - V_3 V_{2,zz} = 0, \tag{5.6}$$

$$V_{3,t} + V_1 V_{3,x} + V_2 V_{3,y} + V_3 V_{3,z} + P_z + B_1(B_{1,z} - B_{3,x})$$

$$+B_2(B_{2,z} - B_{3,,y}) - v_1 V_{3,xx} - v_2 V_{3,yy} - v_3 V_{3,zz} = 0, \tag{5.7}$$

$$B_{1,t} + V_1 B_{1,x} + V_2 B_{1,y} + V_3 B_{1,z} - B_1 V_{1,x} - B_2 V_{1,y} - B_3 V_{1,z}$$

$$-\eta_1 B_{1,xx} - \eta_2 B_{1,yy} - \eta_3 B_{1,zz} = 0\,, \tag{5.8}$$

$$B_{2,t} + V_1 B_{2,x} + V_2 B_{2,y} + V_3 B_{2,z} - B_1 V_{2,x} - B_2 V_{2,y} - B_2 V_{2,z}$$

$$-\eta_1 B_{2,xx} - \eta_2 B_{2,yy} - \eta_3 B_{2,zz} = 0\,, \tag{5.9}$$

$$B_{3,t} + V_1 B_{3,x} + V_2 B_{3,y} + V_3 B_{3,z} - B_1 V_{3,x} - B_2 V_{3,y} - B_3 V_{3,z}$$

$$-\eta_1 B_{3,xx} - \eta_2 B_{3,yy} - \eta_3 B_{3,zz} = 0\,. \tag{5.10}$$

To find the travelling wave solution for equations (5.4)-(5.10), we take the transformation

$$V_i(x,y,z,t) = v_i(\xi), \qquad B_i(x,y,z,t) = b_i(\xi),$$

$$P(x,y,z,t) = p(\xi), \qquad i = 1,2,3 \tag{5.11}$$

where $\xi = x + y + \alpha z + \beta t$;

This reduces equations (5.4)-(5.10) into the following system of ordinary differential equations

$$v_1^/ + v_2^/ + \alpha v_3^/ + b_1^/ + b_2^/ + \alpha b_3^/ = 0, \tag{5.12}$$

$$(\beta + v_1 + v_2 + \alpha v_3) v_1^/ + p^/ + b_2\left(b_2^/ - b_1^/\right) +$$

$$b_3\left(b_3^/ - \alpha b_1^/\right) - (v_1 + v_2 + \alpha^2 v_3) v_1^{//} = 0, \tag{5.13}$$

$$(\beta + v_1 + v_2 + \alpha v_3) v_2^/ + p^/ + b_1\left(b_1^/ - b_2^/\right) +$$

$$b_3\left(b_3^/ - \alpha b_2^/\right) - (v_1 + v_2 + \alpha^2 v_3) v_2^{//} = 0, \tag{5.14}$$

$$(\beta+\ v_1+\ v_2+\ \alpha v_3)v_3^{/}+\ \alpha p^{/}+\ b_1\left(\alpha b_1^{/}-b_3^{/}\right)+$$

$$b_2\left(\alpha b_2^{/}-b_3^{/}\right)-(v_1+\ v_2+\ \alpha^2 v_3)v_3^{//}=0, \tag{5.15}$$

$$(\beta+v_1+v_2+\alpha v_3)b_1^{/}-(b_1+b_2+\alpha b_3)v_1^{/}$$

$$-(\eta_1+\eta_2+\alpha^2\eta_3)b_1^{//}=0\ , \tag{5.16}$$

$$(\beta+v_1+v_2+\alpha v_3)b_2^{/}-(b_1+b_2+\alpha b_3)v_2^{/}$$

$$-(\eta_1+\eta_2+\alpha^2\eta_3)b_2^{//}=0\ , \tag{5.17}$$

$$(\beta+v_1+v_2+\alpha v_3)b_3^{/}-(b_1+b_2+\alpha b_3)v_3^{/}$$

$$-(\eta_1+\eta_2+\alpha^2\eta_3)b_3^{//}=0\ , \tag{5.18}$$

where $^{/}=\frac{d}{d\xi}$ Balancing the highest order of linear terms with nonlinear terms in systems (5.12) - (5.18) suggests the following ansatz

$$v_1=\ \gamma_1+\gamma_2\psi,\quad v_2=\ \gamma_3+\gamma_4\psi,\quad v_3=\ \gamma_5+\gamma_6\psi, \tag{5.19}$$

$$b_1=\ \delta_1+\delta_2\psi,\quad b_2=\ \delta_3+\delta_4\psi,\quad b_3=\ \delta_5+\delta_6\psi, \tag{5.20}$$

$$p=\rho_1+\rho_2\psi+\rho_3\psi^2, \tag{5.21}$$

where $\gamma_i, \delta_i, \rho_i, \alpha$ and $\beta$, are constants to be determined, and the function $\psi$ satisfies the Riccati equation

$$\psi^{/\,2} + \epsilon k^2 \psi^2 = \epsilon k^4, \qquad k \geq 0, \qquad \epsilon = \pm 1, \tag{5.22}$$

There are three kinds of general solutions [134-136]

$$\psi = k \sin k\xi\,, \qquad \text{or} \qquad \psi = k \cos k\xi\,, \qquad \text{when}\;\; \epsilon = 1 \tag{5.23}$$

$$\psi = \text{constant, when} \qquad k = 0, \tag{5,24}$$

and

$$\psi = k \cos\text{h}\, k\xi, \quad \text{when} \qquad \epsilon = -1. \tag{5.25}$$

Substituting (5.19) -(5.21) into system (5.12) -(5.18) and using the Riccati equation (5.22), we obtain

$$(\gamma_2 + \gamma_4 + \alpha\gamma_6)\psi^{/} = (\delta_2 + \delta_4 + \alpha\delta_6)\psi^{/} = 0, \tag{5.26}$$

$$(\gamma_2\beta)\psi^{/} + (\gamma_1 + \gamma_2\psi)(\gamma_2\psi^{/}) + (\gamma_3 + \gamma_4\psi)(\gamma_2\psi^{/})$$

$$+\alpha(\gamma_5 + \gamma_6\psi)(\gamma_2\psi^{/}) + (\rho_2\psi^{/}) + (2\rho_3\psi\psi^{/}) +$$

$$(\delta_3 + \delta_4\psi)(\delta_4 - \delta_2)\psi^{/} + (\delta_5 + \delta_6\psi)(\delta_6 - \alpha\delta_2)\psi^{/}$$

$$-(v_1 + v_2 + \alpha^2 v_3)(\gamma_2\psi^{//}) = 0, \tag{5.27}$$

$$(\beta+\gamma_1+\gamma_2\psi+\gamma_3+\gamma_4\psi+\alpha\gamma_5+\alpha\gamma_6\psi)(\gamma_4\psi^{/})+$$

$$(\rho_2\psi^{/})+(2\rho_3\psi\psi^{/})+(\delta_1+\delta_2\psi)(\delta_2-\delta_4\psi)\psi^{/}+$$

$$(\delta_5+\delta_6\psi)(\delta_6-\alpha\delta_4)\psi^{/}-$$

$$(v_1+v_2+\alpha^2 v_3)(\gamma_4\psi^{//})=0, \tag{5.28}$$

$$(\beta+\gamma_1+\gamma_2\psi+\gamma_3+\gamma_4\psi+\alpha\gamma_5+\alpha\gamma_6\psi)(\gamma_6\psi^{/})+$$

$$(\alpha\rho_2\psi^{/})+(2\alpha\rho_3\psi\psi^{/})+(\delta_1+\delta_2\psi)(\alpha\delta_2-\delta_6)\psi^{/}+$$

$$(\delta_3+\delta_4\psi)(\alpha\delta_4-\delta_6)\psi^{/}-$$

$$(v_1+v_2+\alpha^2 v_3)(\gamma_6\psi^{//})=0, \tag{5.29}$$

$$(\beta+\gamma_1+\gamma_2\psi+\gamma_3+\gamma_4\psi+\alpha\gamma_5+\alpha\gamma_6\psi)(\delta_2\psi^{/})$$

$$-(\delta_1+\delta_2\psi+\delta_3+\delta_4\psi+\alpha\delta_5+\alpha\delta_6\psi)(\gamma_2\psi^{/})+$$

$$-(\eta_1+\eta_2+\alpha^2\eta_3)(\delta_2\psi^{//})=0, \tag{5.30}$$

$$(\beta+\gamma_1+\gamma_2\psi+\gamma_3+\gamma_4\psi+\alpha\gamma_5+\alpha\gamma_6\psi)(\delta_4\psi^{/})$$

$$-(\delta_1+\delta_2\psi+\delta_3+\delta_4\psi+\alpha\delta_5+\alpha\delta_6\psi)(\gamma_4\psi^{/})+$$

$$-(\eta_1+\eta_2+\alpha^2\eta_3)\left(\delta_4\psi^{//}\right)=0, \tag{5.31}$$

$$(\beta+\gamma_1+\gamma_2\psi+\gamma_3+\gamma_4\psi+\alpha\gamma_5+\alpha\gamma_6\psi)(\delta_6\psi^{/})$$

$-(\delta_1 + \delta_2\psi + \delta_3 + \delta_4\psi + \alpha\delta_5 + \alpha\delta_6\psi)(\gamma_6\psi^{/}) +$

$$-(\eta_1 + \eta_2 + \alpha^2\eta_3)\big(\delta_6\psi^{//}\big) = 0, \tag{5.32}$$

By setting the coefficients of all powers of $\psi, \psi^{/}$ $and$ $\psi\psi^{/}$ to zero, we get a set of algebraic equations for the variables $\gamma_i, \delta_i, \rho_i, \alpha$ and $\beta$.

$$(\gamma_2 + \gamma_4 + \alpha\gamma_6) = 0, \quad (\delta_2 + \delta_4 + \alpha\delta_6) = 0, \tag{5.33}$$

$$(v_1 + v_2 + \alpha^2 v_3)(\gamma_2 k^2) = 0, \tag{5.34}$$

$\gamma_2\beta + \gamma_1\gamma_2 + \gamma_3\gamma_2 + \alpha\gamma_5\gamma_2 + \rho_2$

$$+\delta_3\delta_4 - \delta_3\delta_2 + \delta_5\delta_6 - \alpha\delta_5\delta_2 = 0, \tag{5.35}$$

$$\gamma_4^2 + \gamma_2\gamma_4 + \alpha\gamma_6\gamma_2 + 2\rho_3 + \delta_4^2 - \delta_4\delta_2 + \delta_6^2 - \alpha\delta_6\delta_2 = 0, \tag{5.36}$$

$$(v_1 + v_2 + \alpha^2 v_3)(\gamma_4 k^2) = 0, \tag{5.37}$$

$\gamma_4\beta + \gamma_1\gamma_4 + \gamma_3\gamma_4 + \alpha\gamma_5\gamma_4 + \rho_2$

$$+\delta_1\delta_2 - \delta_1\delta_4 + \delta_5\delta_6 - \alpha\delta_5\delta_4 = 0, \tag{5.38}$$

$$\gamma_4^2 + \gamma_2\gamma_4 + \alpha\gamma_4\gamma_6 + 2\rho_3 + \delta_2^2 - \delta_4\delta_2 + \delta_6^2 - \alpha\delta_6\delta_2 = 0, \tag{5.39}$$

$$(v_1 + v_2 + \alpha^2 v_3)(\gamma_6 k^2) = 0, \tag{5.40}$$

$\gamma_6\beta + \gamma_1\gamma_6 + \gamma_3\gamma_6 + \alpha\gamma_5\gamma_6 + \alpha\rho_2$

$$+\alpha\delta_1\delta_2 - \delta_1\delta_6 + \delta_3\delta_4 - \alpha\delta_3\delta_6 = 0, \qquad (5.41)$$

$$\alpha\gamma_6^2 + \gamma_2\gamma_6 + \gamma_4\gamma_6 + 2\alpha\rho_3$$

$$+\alpha\delta_2^2 - \delta_6\delta_2 + \alpha\delta_4^2 - \delta_6\delta_2 = 0, \qquad (5.42)$$

$$(\eta_1 + \eta_2 + \alpha^2\eta_3)(\delta_2 k^2) = 0, \qquad (5.43)$$

$$\delta_2\beta + \gamma_1\delta_2 + \gamma_3\delta_2 + \alpha\gamma_5\delta_2 - \gamma_2\delta_1 - \gamma_2\delta_3 - \alpha\delta_5\gamma_2 = 0, \qquad (5.44)$$

$$\gamma_2\delta_2 + \gamma_4\delta_2 + \alpha\gamma_6\delta_2 - \gamma_2\delta_2 - \delta_4\gamma_2 - \alpha\delta_6\gamma_2 = 0, \qquad (5.45)$$

$$(\eta_1 + \eta_2 + \alpha^2\eta_3)(\delta_4 k^2) = 0, \qquad (5.46)$$

$$\delta_4\beta + \gamma_1\delta_4 + \gamma_3\delta_4 + \alpha\gamma_5\delta_4 - \gamma_4\delta_1 - \gamma_4\delta_3 - \alpha\delta_5\gamma_4 = 0, \qquad (5.47)$$

$$\gamma_2\delta_4 + \gamma_4\delta_4 + \alpha\gamma_6\delta_4 - \gamma_4\delta_2 - \delta_4\gamma_4 - \alpha\delta_6\gamma_4 = 0, \qquad (5.48)$$

$$(\eta_1 + \eta_2 + \alpha^2\eta_3)(\delta_6 k^2) = 0, \qquad (5.49)$$

$$\delta_6\beta + \gamma_1\delta_6 + \gamma_3\delta_6 + \alpha\gamma_5\delta_6 - \gamma_6\delta_1 - \gamma_6\delta_3 - \alpha\delta_5\gamma_6 = 0, \qquad (5.50)$$

$$\gamma_2\delta_6 + \gamma_4\delta_6 + \alpha\gamma_6\delta_6 - \gamma_6\delta_2 - \delta_4\gamma_6 - \alpha\delta_6\gamma_6 = 0, \qquad (5.51)$$

The solutions for these equations were generated using the symbolic software package Mathematica,

$$v_1 = v_2 = -2v_3, \qquad \eta_1 = \eta_2 = -2\eta_3,$$

$$\alpha = 2, \qquad \gamma_2 = \gamma_4 = \delta_2 = \delta_4 = -\gamma_6 = -\delta_6 = d_0,$$

$$\rho_3 = \frac{-3}{2} d_0^2, \qquad \rho_2 = d_0(3 - c_0), \qquad \beta = c_0 - a_0, \tag{5.52}$$

where

$a_0 = \gamma_1 + \gamma_3 + 2\gamma_5, \quad c_0 = \delta_1 + \delta_3 + 2\delta_5,$

and $a_0, c_0, \rho_2, \rho_3$, and k are arbitrary constants. Since $k$ is an arbitrary parameter, according to (5.19)-(5.21), (5.23)-(5.25) and (5.52), we obtain three kinds of travelling wave solutions for the new coupled MHD system (5.1)-(5.3), namely,

(1) a periodic solution with $\epsilon = 1$

$$V_1 = \gamma_1 + k d_0 \sin k\xi, \qquad V_2 = \gamma_3 + k d_0 \sin k\xi,$$

$$V_3 = \gamma_5 - k d_0 \sin k\xi, \tag{5.53}$$

$$B_1 = \delta_1 + k d_0 \sin k\xi, \qquad B_2 = \delta_3 + k d_0 \sin k\xi,$$

$$B_3 = \delta_5 - k d_0 \sin k\xi, \tag{5.54}$$

$$P = \rho_1 + k d_0 (3 - c_0) \sin k\xi - \frac{3}{2} d_0^2 k^2 \sin^2 k\xi, \tag{5.55}$$

(2) a soliton solution with $\epsilon = -1$

$$V_1 = \gamma_1 + kd_0 \cos\mathrm{h}\, k\xi, \qquad V_2 = \gamma_3 + kd_0 \cos\mathrm{h}\, k\xi,$$

$$V_3 = \gamma_5 - kd_0 \cos\mathrm{h}\, k\xi, \tag{5.56}$$

$$B_1 = \delta_1 + kd_0 \cos\mathrm{h}\, k\xi, \qquad B_2 = \delta_3 + kd_0 \cos\mathrm{h}\, k\xi,$$

$$B_3 = \delta_5 - kd_0 \cos\mathrm{h}\, k\xi, \tag{5.57}$$

$$P = \rho_1 + kd_0(3 - c_0) \cos\mathrm{h}\, k\xi - \frac{3}{2} d_0^2 k^2 \cos\mathrm{h}^2\, k\xi, \tag{5.58}$$

(3) a constant solution with $k = 0$k

$$V_1 = \gamma_1, \qquad V_2 = \gamma_3, \qquad V_3 = \gamma_5, \tag{5.59}$$

$$B_1 = \delta_1, \qquad B_2 = \delta_3, \qquad B_3 = \delta_5, \tag{5.60}$$

$$P = \rho_1, \tag{5.61}$$

where $\xi = x + y + 2z + (c_0 + a_0)t$.

The MHD equations govern the dynamics of the velocity and magnetic field in electrically conducting fluids and reflect the basic laws of conservation in physics. These equations can be implemented to study various problems in plasma physics, liquid metals, saltwater, and astrophysics.

The MHD equations involve coupling between the incompressible Navier-Stokes equations (when the magnetic field $\underline{B}$ is identically equal to $0$) governing the fluid and incompressible Euler equations for

$$\boldsymbol{B} = (0, v_1, v_2, v_3) = 0$$

This chapter examines the soliton solutions for the three-dimensional incompressible MHD equations with only magnetic diffusion (without velocity dissipation). The magnetic field, which is present everywhere in the universe, generates a magnetic force and this force influences the dynamics of a moving electrically conducting fluid, potentially changing the geometry or strength of the magnetic field itself. It has been found that the difference in the phase may occur between speed and fluctuations of the magnetic field when the kinetic and magnetic Reynolds numbers are very large. Since the speed and fluctuations of the magnetic field are both circularly polarized, the phase difference makes them no longer parallel or anti-parallel like that in the incompressible MHD.

## 5.3 Conclusions

This chapter presents a stabilized exact soliton solution for the incompressible MHD equations. These stabilized soliton solutions are focused at incompressible fluids and the main technological applications in mind are those related with material processing techniques.

The flow considered here is incompressible and parallel to the magnetic field. Several classes of soliton solutions are obtained in three-dimensional Cartesian coordinates. Previously, Neukirch [72] obtained self-consistent three-dimensional exact solutions to the MHD equations and solved the basic nonlinear equations in terms of Jacobi elliptic functions. Petrie and Neukirch [73] used a transformation method to study the equilibria of MHD equations and solved one component of the problem of the vanishing magnetic field. Petrie et al. [74] obtained two-dimensional exact solutions of the MHD equations with application to solar prominences. Here, we obtained three-dimensional exact solutions in the presence of mass flow and with all three components of the magnetic field.

The three kinds of travelling wave solutions for the new coupled MHD system include a periodic solution with $= 1$ , a soliton solution with $\epsilon = -1$ and a constant solution with $k = 0$.

# Chapter 6
# Analytical solutions for isothermal magnetostatic atmospheres

## 6.1 Introduction

The Sun is an object of great beauty and fascination that has been studied with interest for thousands of years. During this century it has gradually become clear that much of the observed structure owes its existence to the Sun's magnetic field. The effects of the Sun's magnetic field in the solar atmosphere are pervasive and give rise to several anomalies throughout the solar atmosphere. On the photosphere, sunspots can be seen with the naked eye. During a total eclipse and with the aid of a coronagraph, wide arches and helmet streamers can be seen in the corona. Also, high-resolution observations of the Sun from the ground have revealed new features of the photosphere and chromosphere in fine detail. These have been complemented by satellite observations of the transition region and corona directly. The visible solar atmosphere consists of three regions with different physical properties. The lowest is an extremely thin layer of plasma, called the *photosphere*, which is relatively dense and opaque and emits most of the solar radiation. Above it lies the rarer and more transparent *chromosphere*, while the *corona* extends from the top of a narrow *transition region* to the Earth and beyond. Hydrogen is almost wholly ionised in the upper chromosphere, but neutral hydrogen is important in the lower chromosphere and photosphere [92-97].

The Sun is our nearest star. It is important for astronomy because many phenomena which can only be studied indirectly in other stars can be directly observed in the Sun (e.g. stellar rotation, starspots, the structure of the stellar surface). Our present picture of the Sun is based both on observations and on theoretical calculations. Some observations of the Sun disagree with the theoretical solar models. The details of the models will have to be changed, but the general picture should remain valid.

Solar MHD is an important tool for understanding many solar phenomena.

It also plays a crucial role in explaining the behaviour of more general cosmical magnetic fields and plasmas, since the Sun provides a natural laboratory in which such behaviour may be studied. While terrestrial experiments are invaluable in demonstrating general plasma properties, conclusions from them cannot be applied uncritically to solar plasmas and have in the past given rise to misconceptions about solar magnetic field behaviour.

The magnetostatic equations were widely used to model the magnetic structure of the solar atmosphere [41]. An investigation of a family of isothermal magnetostatic atmospheres with one ignorable coordinate corresponding to a uniform gravitational field in a plane geometry is carried out. The balance of force consists of the force between $\boldsymbol{J} \wedge \boldsymbol{B}$ ($\boldsymbol{B}$ is the induction of the magnetic field, $\boldsymbol{J}$ is the strength of the electrical current), gravitational force, and gradient force of the gas pressure. However, in many models, the temperature distribution is defined a priori and any direct references to the energy equations are not taken into consideration. Magnetostatic equations in solar physics have been used to model many different phenomena, like the slow growth of solar flares or the magnetostatic support of prominence [43]. The nonlinear equilibrium problem for modelling astrophysical plasmas was resolved in several cases [44].

In this chapter, the problem of a plasma equilibrium in a gravitational field is identified. For the case of an isothermal atmosphere, the empirical nonlinear periodic solutions of the corresponding elliptic equation are provided within a uniform gravitational field for various arbitrary function choices. Such solutions are derived using the sine–cosine and Bäcklund transformations.

## 6.2 Basic equations

The relevant magnetohydrostatic equations consist of the equilibrium

equation [46, 47]:

$$\boldsymbol{J}\, \Lambda\, \boldsymbol{B} - \rho\nabla\Phi - \nabla P = 0 \tag{6.1}$$

and Maxwell's equations:

$$\boldsymbol{J} = \frac{\boldsymbol{\nabla}\, \Lambda\, \boldsymbol{B}}{\mu}, \tag{6.2}$$

$$\boldsymbol{\nabla} \,.\, \boldsymbol{B} = \boldsymbol{0}, \tag{6.3}$$

where $P, \rho$ , $\mu = 4\pi$ and $\Phi$ are the gas pressure, the mass density, the magnetic permeability and the gravitational potential, respectively. It is assumed that the temperature is uniform in space and that the plasma behaves as an ideal gas satisfying the equation of state $p = \rho\, R_0 T_0$, where $R_0$ is the gas constant and $T_0$ is the temperature. The magnetic field $\boldsymbol{B}$ can be written as,

$$\boldsymbol{B} = \boldsymbol{\nabla}\, u(y,z)\; \Lambda\; \boldsymbol{e_x} + B_x\, \boldsymbol{e_x} = \left(B_x, \frac{\partial u(y,z)}{\partial z}, \frac{-\partial u(y,z)}{\partial y}\right), \tag{6.4}$$

where $u\,(y,z)$, $B_x\,(y,z)$ are the magnetic flux function and $x$ −component of $B$ . The form of (6.4) for B ensures that $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, and there is no mono pole or defect structure. Equation (6.1) requires the pressure and density be of the form [46]:

$$P(y.z) = P\big(u(y,z)\big)e^{\frac{-z}{h}}, \qquad \rho(y,z) = \frac{1}{(gh)}P\big(u(y,z)\big)e^{\frac{-z}{h}} \tag{6.5}$$

where $h = \frac{R_O T_O}{g}$ is the scale height. Substituting equations (6.2) – (6.5) in equation (6.1), we obtain

$$\nabla^2 u + f(u)e^{\frac{-z}{h}} = 0 \tag{6.6}$$

where

$$f(u) = 4\,\pi \frac{dP}{du}. \tag{6.7}$$

Equation (6.7) gives

$$P(u) = P_0 + \frac{1}{4\pi}\int f(u)du \tag{6.8}$$

Substituting equation (6.8) into equation (6.5), we obtain

$$P(y,z) = (P_0 + \frac{1}{4\,\pi}\int f(u)du\,)\,e^{\frac{-z}{h}}, \tag{6.9}$$

$$\rho(y,z) = \frac{1}{(gh)}(P_0 + \frac{1}{4\,\pi}\int f(u)du)e^{\frac{-z}{h}}, \tag{6.10}$$

where $P_0$ is constant. Taking the transformations

$$x_1 = e^{\frac{-z}{l}}\cos\frac{y}{l}, \quad x_2 = e^{\frac{-z}{l}}\sin\frac{y}{l}, \quad x_1 + i\,x_2 = e^{\frac{-z}{l}}e^{\frac{iy}{l}} \tag{6.11}$$

reduces (6.6) to

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial\, x_2^2} + l^2 f(u)e^{\left(\frac{2}{l}-\frac{1}{h}\right)z} = 0 \tag{6.12}$$

The above equations have been given in Khater et al. [43-47].

### 6.3 Liouville equation

We will study the Liouville equation as special case of equation (6.12). Let us assume $f(u)$ has the form [62]:

$$f(u) = -\alpha^2 u_0 e^{-\frac{u}{u_0}}, \tag{6.13}$$

where $u_0$ and $\alpha^2$ are constants. Hence

$$P(y,z) = \left(P_0 + \frac{\alpha^2 u_0^2}{8\,\pi}e^{-\frac{2\,u}{u_0}}\right)e^{\frac{-z}{h}}. \tag{6.14}$$

Inserting equation (6.13) into equation (6.12), we obtain

$$\nabla^2\ u/u_0 = l^2\alpha^2 e^{\frac{-2\,u}{u_0}+\left(\frac{2}{l}-\frac{1}{h}\right)z}\ , \tag{6.15}$$

where $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$,

Let us set

$$\frac{u}{u_0} = \frac{z}{L} + \phi(y,z), \tag{6.16}$$

where $L$ is a constant. Then, equation (6.15) becomes

$$\nabla^2\,\phi = l^2\alpha^2 e^{-2\,\phi-\left(\frac{2}{L}+\frac{1}{h}-\frac{2}{l}\right)z}. \tag{6.17}$$

If $l$ is given by

$$\frac{2}{l} = \frac{2}{L} + \frac{1}{h}, \tag{6.18}$$

then inserting this into equation (6.17), we obtain a Liouville type equation

$$\emptyset_{xx} + \emptyset_{tt} - \alpha^2 l^2 e^{-2\emptyset} = 0. \tag{6.19}$$

In order to apply the sine-cosine method, we first use the wave variable $\xi = x - k\,t$ where $k$ is a constant, to carry the PDE in two independent variables (5.19), into the following ordinary differential equation

$$( 1+k^2)\, \phi^{//} \;=\; \alpha^2 l^2 e^{-2\emptyset}. \tag{6.20}$$

Now we use the following transformation to find a solution of Liouville's equation (6.20). If we let

$$v = e^{-2\emptyset}, \tag{6.21}$$

then equation (6.20) becomes

$$v\, v^{//} - \left(v^{/}\right)^2 \;+\; b\;\, v^3 \;\; = 0, \qquad where \qquad b = \frac{2\, \boldsymbol{\alpha^2 l^2}}{(\,1+k^2)} \tag{6.22}$$

$$v(\xi) = \;a\;\; sin^{\,n}\,(\mu\,\xi\,), \tag{6.23}$$

$$v^{/}(\xi) = a\;\;\mu\, n\;\; sin^{\,n-1}\,(\mu\,\xi\,) \cos\,(\mu\,\xi\,), \tag{6.24}$$

$$v^{//}(\xi) = -\;\, a\,\mu^2\, n^2\;\, sin^{\,n}\,(\mu\,\xi\,) + \;a\,\mu^2\, n\,(n-1)\, sin^{\,n-2}\,(\mu\xi\,). \tag{6.25}$$

Substituting (6.23) – (6.25) into (6.22) yields

$$a^2\, \mu^2 [n(n-1) - \;n^2]\, sin^{\,2\,n-2}\,(\mu\,\xi\,) \;+ b\; a^3\, sin^{\,3\,n}\,(\mu\,\xi\,) = 0. \tag{6.25}$$

Equating the exponents and the coefficients of each pair of the sine functions, we find the following system of algebraic equations:

$$3n = 2\,n - 2, \qquad b\; a \;+ 2\,\mu^2 \;= 0, \qquad n \neq 0, \tag{6.26}$$

From system (6.26), we obtain

$$n = -2, \qquad \mu = \alpha\, l \sqrt{\frac{-a}{1+k^2}}\,. \tag{6.27}$$

which leads to the following formal solitary wave solutions to equation (6.22)

$$v(x,t) = a \ \operatorname{csch}^2\left(\alpha\, l \sqrt{\frac{a}{1+k^2}}\,(x - k\ t)\right), \qquad a > 0 \tag{6.28}$$

From this we obtain the solution to equation (6.19)

$$\phi(x,t) = \frac{-1}{2}\, ln[a \ \operatorname{csch}^2\left(\alpha\, l \sqrt{\frac{a}{1+k^2}}\,(x - k\ t)\right)], \qquad a > 0 \tag{6.29}$$

This is a class of exact analytic solutions to the nonlinear Liouville equation (6.19).

Figures 6.1 and 6.2 can be employed to describe isothermal magnetostatic atmosphere. When the parameters take on special values ($\alpha = 2, l = 3$, $\mathrm{k} = 1$ and $\mathrm{a} = 2$), the solitary wave solutions are obtained in the from of hyperbolic function. These exact solutions include the hyperbolic function solutions and trigonometric function solutions.

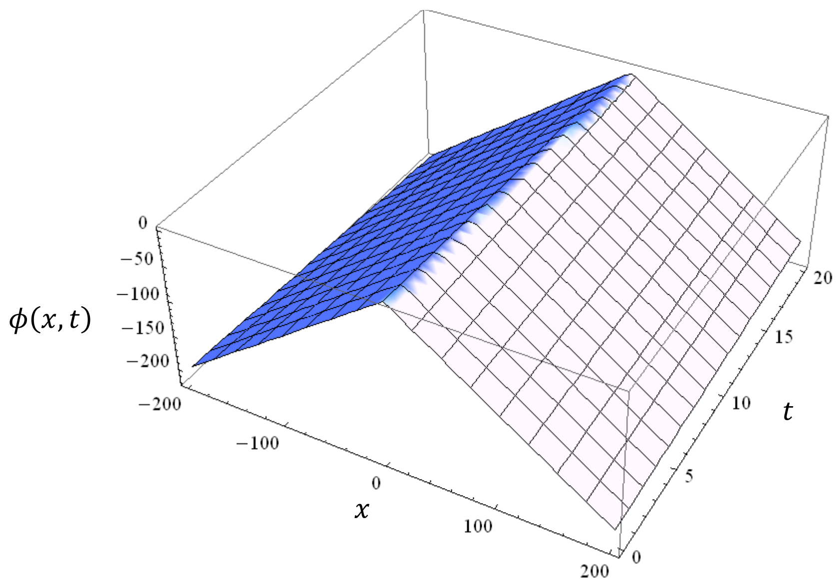



Figure 6.1: Th wave solution of Liouville's equation with $\alpha = 2, l = 3$, $\mathrm{k} = 1$ and $\mathrm{a} = 2$

In the Figure 6.1, we have shown the wave solution results of Liouville's equation do indeed correspond to a completely integrable dynamical system. This system describes the isothermal magnetostatic atmosphere, and the construction of examples of such interactions even in three - dimensional space-time is very helpful for studying general theoretical problems.

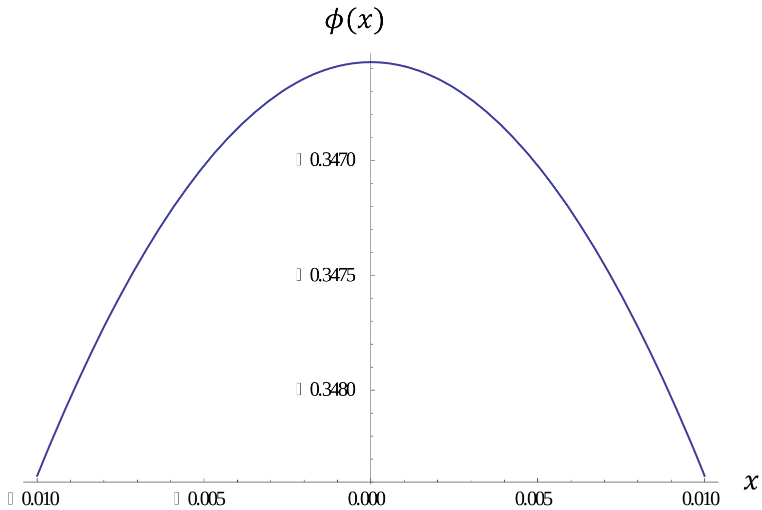



Figure 6.2: Travelling wave solutions of eq. (6.29) with $t = 0$,

$\alpha = 2, l = 3,$ k = 1 and a = 2

In Figure 6.2, our result by no means contradicts the numerous formulations of the no-interaction theorem for relativistic particles, since in these formulations it is always assumed that the particle coordinates $x$ are included among the canonical variables.

*or*

$$\phi(y,z) = \frac{-1}{2}\, ln[a \ \ \text{csch}^2\left(\alpha\, l \ \sqrt{\frac{a}{1+k^2}}\ e^{\frac{-z}{l}}(\cos\frac{y}{l} - k \ \ \sin\frac{y}{l})\right)], (6.30)$$

The exact results (6.30) are compared with the numerical results [41-47]. The sine – cosine method has been successfully used to obtain some exact travelling wave solutions for the Liouville equation (see figures 6.3 and 6.4).

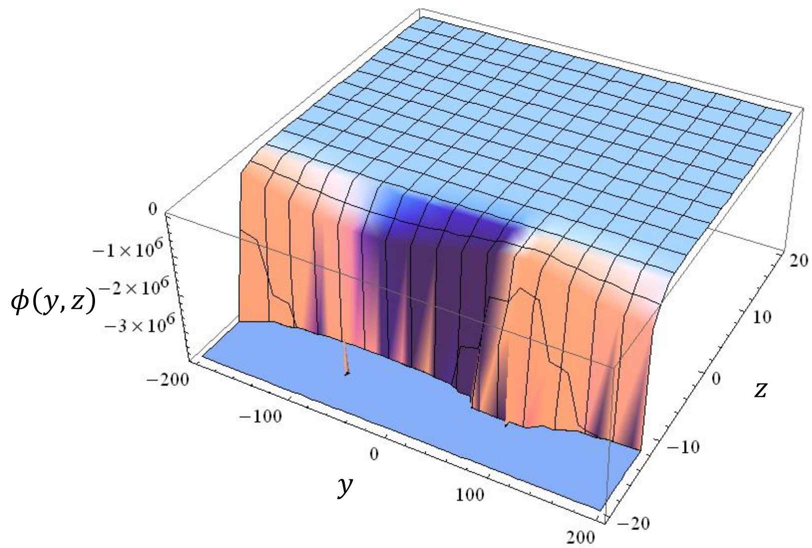



Figure 6.3: Travelling wave solution of eq. (6.30) with $\alpha = 2, l = 1$, $\mathrm{k} = 1$ and $\mathrm{a} = 2$

Figure 6.3 shows the exact solution and travelling wave solution of eq. (6.30) for specific values of $\alpha, l, k$ $and$ $a$. From the given graph, it can be observed that both the exact solution and numerical solution are in strong agreement with each other.

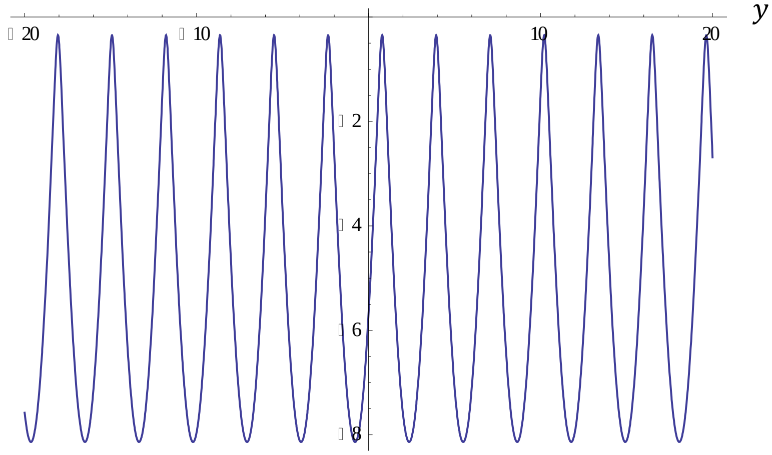



Figure 6.4: Travelling wave solution of eq. (6.30) with $z = 0$,

$\alpha = 2, l = 3, \quad \mathrm{k} = 1$ and $\mathrm{a} = 2$

Figure 6.4 show the wave solution resulting from eq. (6.30) for various values of the distance term $y$, and for specific values of $z, \alpha, l, k$ and $a$.

When $P_0 = 0,$ then from (6.14) the plasma pressure is

$$P(y,z) = \frac{a\,\alpha^2 u_0^2}{8\,\pi} e^{\left(\frac{-2\,z}{l} - \frac{z}{h}\right)} \operatorname{csch}^2\left(\alpha\, l\, \sqrt{\frac{a}{1+k^2}}\; e^{\frac{-z}{l}} \left(\cos\frac{y}{l} - k\ \sin\frac{y}{l}\right)\right) \tag{6.31}$$

The results for the pressure (6.31) are shown in Figures 6.5 & 6.6. These figures show the plasma pressure of the models calculated with different values of the parameter( $\alpha, l, \mathrm{a}, \mathrm{k}$ ) , the rest of the parameters being the same.

The analytical solutions presented describe isothermal magnetostatic atmospheres in uniform gravity and varying in three Cartesian dimensions.

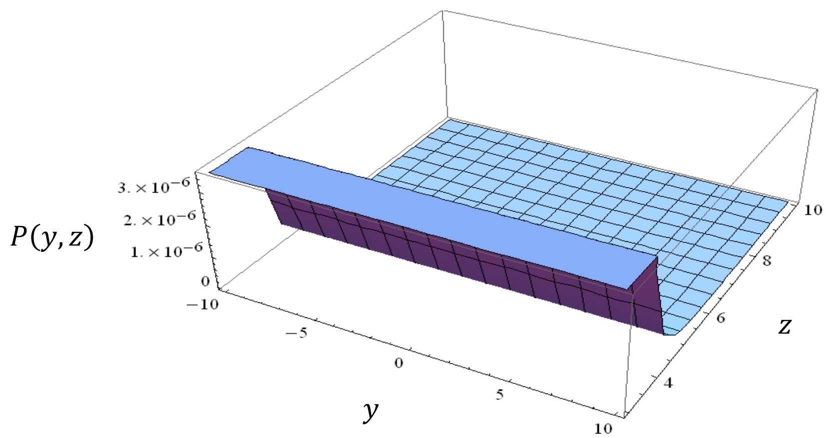



Figure 6.5: The graph of plasma pressure $P(y,z)$

with $\alpha = 2, l = 2,$ $h = 1,$ $u_0 = 1,$ $\mathrm{k} = 1$ and $\mathrm{a} = 2$

Figure 6.5 shows that the plasma pressure depends on many parameters ( $\alpha$, $h, l,$ a, k ). The solutions are adequate to describe an isothermal atmosphere in a uniform gravitational field showing parallel filaments of diffuse, magnetized plasma suspended horizontally in equilibrium.

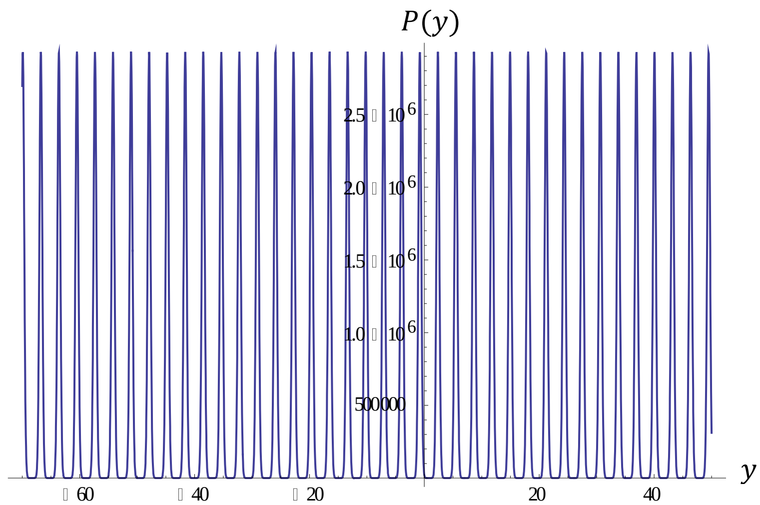



Figure 6.6: The graph of plasma pressure $P(y, z)$

with $z = 0$, $\alpha = 2, l = 3$, $u_0 = 1$, $\mathrm{k} = 1$ and $\mathrm{a} = 2$

In Figure 6.6 shows the plasma pressure of eq. (6.31) for various values of the distance term $y$ and the variable $z = 0$ for special values of the parameters $\alpha$, $h, l$, $a$, $k$ .

### 6.4 Sinh – Poisson equation

In this section, we find the sinh – Poisson equation which plays an important role in the soliton solutions model. This equation will be a special case of equation (6.12). If we assume

$$f(u) = -\frac{l}{4}\frac{u_0}{h}\sinh\psi, \qquad \text{where} \qquad \psi = \frac{u}{h\,u_0} \tag{6.32}$$

then from equations (6.9) and (6.33), we obtain

$$P(y,z) = \left(P_0 - \frac{l\,u_0^2}{8\,\pi} cosh\,\psi\right) e^{\frac{-z}{h}}. \tag{6.33}$$

Substituting (6.32) into (6.12), we get

$$\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial\, x_2^2} = l\;\; sinh\,\psi, \qquad where \quad l = 2h. \tag{6.34}$$

Equation (6.34) is a well-known equation and under the transformation

$$x = \frac{\sqrt{l}}{2}\;(\,x_1 + i\,x_2), \quad t = \frac{\sqrt{l}}{2}\;(\,x_1 - i\,x_2), \;\; U(x,t) = \;\psi(x_1, x_2), \tag{6.35}$$

we obtain the sinh – Poisson equation in the form

$$U_{x\,t} = sinh\;(\;U\;), \tag{6.36}$$

We use the B$\ddot{a}$cklund Transformations (BTs) approach to analytically solve equation (6.36). The BTs technique is one of the direct methods of generating a new nonlinear solution from a known solution (see, for example, [42]). Earlier, Konno and Wadati [48] had derived certain BTs for the Abowitz, Kaup, Newell and Segur (AKNS) class of Nonlinear Evolution

Equations (NLEEs). Informally, a BT is defined as a system of equations relating one solution of a given equation to another solution of the same equation, possibly with different values of the parameters, or to a solution of another equation. Thus, the problem of obtaining new solutions by BTs is equivalent to obtaining the wave function.

It is known that many NEEs can be derived from the AKNS system

$$\Phi_x = P\Phi, \qquad \Phi_t = Q\Phi, \tag{6.37}$$

where

$$\Phi = \begin{pmatrix} \emptyset_1 \\ \emptyset_2 \end{pmatrix}, \tag{6.38}$$

and P and Q are the $2 \times 2$ null-trace matrices

$$P = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix}, \qquad Q = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{6.39}$$

Here η is a parameter which is independent of $x$ and $t$ while q and $r$ are functions of $x$ and $t$; $P$ and $Q$ must satisfy the integrability condition

$$P_t - Q_x + PQ - Q\,P = 0, \tag{6.40}$$

or, in component form

$$-A_x + qC - rB = 0, \tag{6.41}$$

$$q_t - B_x + 2\eta B - 2qA = 0, \tag{6.42}$$

$$r_t - C_x + 2rA - 2\eta C = 0. \tag{6.42}$$

By a suitable choice of $r$, $A$, $B$, and C in (6.39) we can obtain NLEE which $q$ must satisfy. Konno and Wadati [14] introduced the functions

$$\Gamma = \phi_1/\phi_2 \qquad and \qquad - \tag{6.43}$$

$$U' = U + f(\Gamma, \eta), \tag{6.44}$$

where $U'$ is the new solution created from the old solution $U$.

From the above sinh-Poisson equation (6.36)

$$P = \begin{pmatrix} \eta & \frac{1}{2}U_x \\ \frac{1}{2}U_x & -\eta \end{pmatrix}, \qquad Q = \frac{1}{4\eta}\begin{pmatrix} \cosh U & -\sinh U \\ \sinh U & -\cosh U \end{pmatrix}, \tag{6.45}$$

$$\Gamma = \phi_1/\phi_2\ , \tag{6.46}$$

$$U' = U - 4\tanh^{-1}\Gamma. \tag{6.47}$$

Now we select a known solution of the above NLEE and substitute this solution into the corresponding matrices $P$ and $Q$. Next, we solve Eq (6.40) for $\phi_1$ and $\phi_2$. Then, by (6.47) and the corresponding BT, we obtain a new solution of NEE [87].

The known solution is a constant $U = U_0$

Substituting $U_0 = n\,\pi\,i, \quad n = 0, \pm 1, \pm 2, \cdots$ into the matrices P and Q in (6.45), then by (6.37), we have

$$d\phi = \phi_x\,dx + \phi_t\,dt = P\,\phi\;d\,\theta, \qquad \theta = x - k\,t, \tag{6.48}$$

where

$$P = \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}, \qquad \theta = x - k\,t, \quad k = \frac{(-1)^{n-1}}{4\eta^2}. \tag{6.49}$$

The solution of equation (6.48) is

$$\phi = e^{\theta\,P}\phi_0 = \left( I + \frac{\theta\,P}{1\,!} + \frac{\theta^2\;P^2}{2\;!} + \frac{\theta^3\;P^3}{3\;!} + \cdots \right) \phi_0, \tag{6.50}$$

where $\phi_0$ is a constant column vector. The solution of equation (6.50) is

$$\phi = \begin{pmatrix} \cosh\eta\,\theta + \sinh\eta\,\theta & 0 \\ 0 & \cosh\eta\,\theta - \sinh\eta\,\theta \end{pmatrix} \phi_0. \tag{6.51}$$

Now, we choose $\phi_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in (6.51), and get

$$\phi = \begin{pmatrix} e^{\eta\,\theta} \\ e^{-\eta\,\theta} \end{pmatrix}, \tag{6.52}$$

Substitute (6.52) into (6.46), then by (6.47), we obtain the new solution of the sinh-Poisson equation (6.36)

$$U' = n\pi\, i - \; 4\tanh^{-1}\left(e^{\,2\,\eta\,\theta}\right), \qquad \theta = x - \frac{(-1)^{n-1}}{4\eta^2}\,t, \tag{6.53}$$

when $n = 0$

$$U'(x,t) = -\,4\tanh^{-1}\left(e^{\,2\,\eta\,\theta}\right), \qquad \theta = x + \frac{1}{4\,\eta^2}\,t. \tag{6.54}$$

hence

$$\psi(x_1,x_2) = -\,4\tanh^{-1}\left(e^{\,2\,\eta\,\theta}\right), \;\; \theta = \frac{\sqrt{l}}{2}\,[\left(1+\frac{1}{4\,\eta^2}\right)x_1 + i\left(1-\frac{1}{4\,\eta^2}\right)x_2]$$

(6.55)

We have succeeded in establishing a method of generating a large class of exact periodic solutions for the sinh-Poisson equation which describe the equilibrium states of a two-dimensional guiding center plasma or a two-dimensional line vortex systems in fluids. Figs. (6.7) and (6.8) show how the solution properties of the three modes change when $\eta = \frac{1}{2}$ is varied while keeping other parameters unchanged.

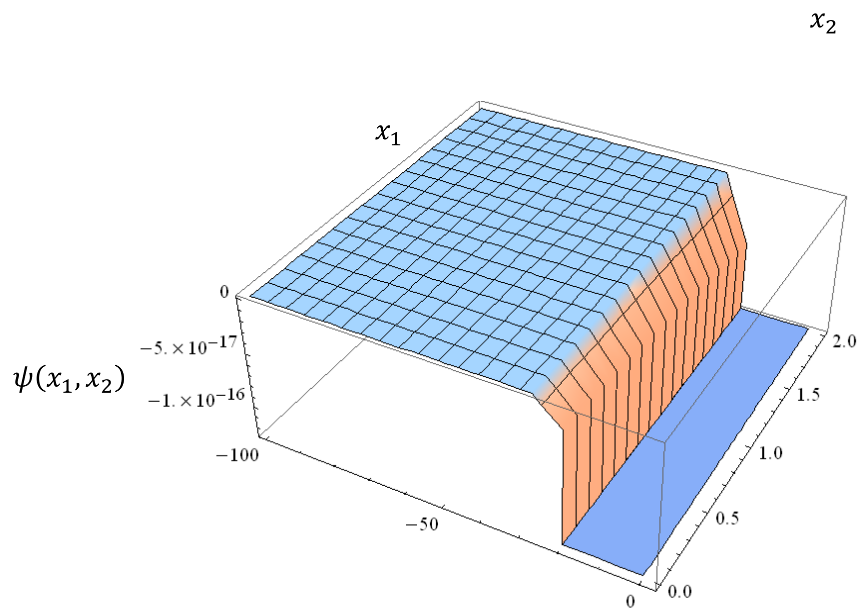



Figure 6.7: The wave solution of $\text{sinh} - \text{ Poisson equation with } \eta = \frac{1}{2}$

The sinh-Poisson equation describes a stream function configuration of a stationary two-dimensional (2D) Euler flow.

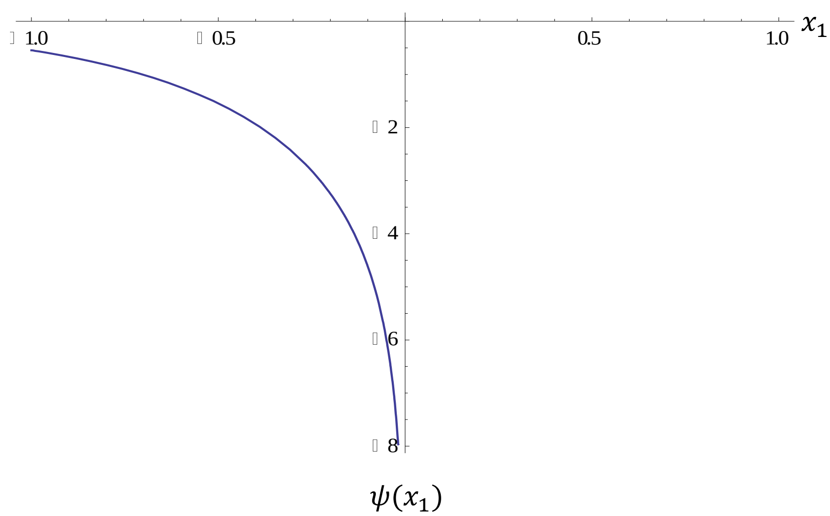



Figure 6.8: Travelling wave solution of eq. (6.55) with $t = 0$

In Figure 6.8, the wave solution of the eq. (6.55) is displayed for various values of the variables $x_1$ and $x_2 = 0$, and for the special case of $\eta = \frac{1}{2}$.

When $\eta = \frac{i}{2}$, we have $2\,\eta\,\theta = i\,\theta = i\,(x-t) = \sqrt{l}\;x_2 = \sqrt{l}\,e^{\frac{-z}{l}}\,sin\,\frac{y}{l}$

The new solution of the sinh – Poisson equation (6.36) is

$$\psi(y,z) = -\,4\tanh^{-1}\left(e^{\sqrt{l}\,e^{\frac{-z}{l}}\,sin\frac{y}{l}}\right), \qquad (6.56)$$

Figure 6.9 shows a surface plot of the exact solution $(6.56)$ *of sinh – Poisson equation* at $l = 1$. The Bäcklund transformations method has been successfully used to obtain exact travelling wave solutions for the Sinh-Poisson equation. Figure 6.10 shows the same solution line plots as in Figure 6.9. The parameter values used in this plot are $z = 0\,, l = 1$

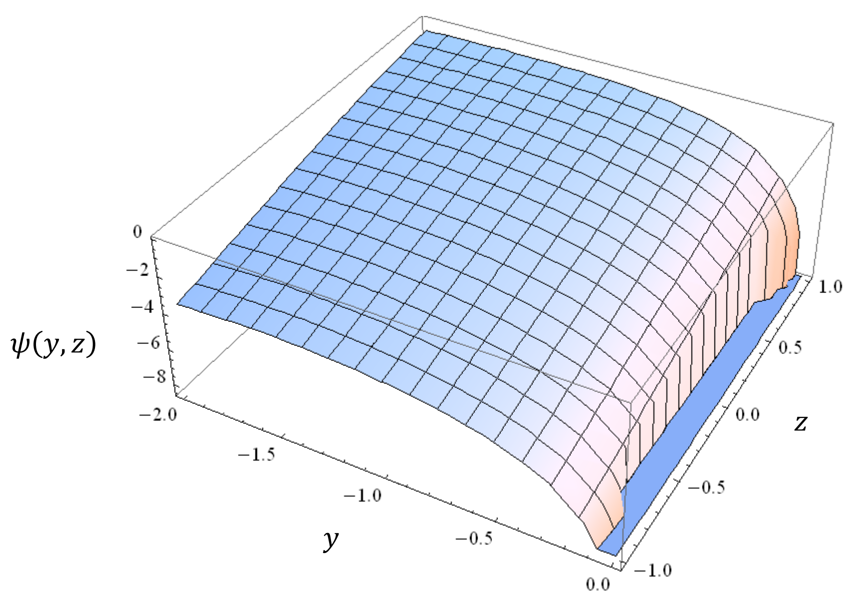



Figure 6.9: Travelling wave solution of Eq. (6.56) with $l = 1$

Figure 6.9 shows the three dimensional plot of the exact solution $(6.56)$ of sinh – Poisson equation for the special case of $l = 1$.

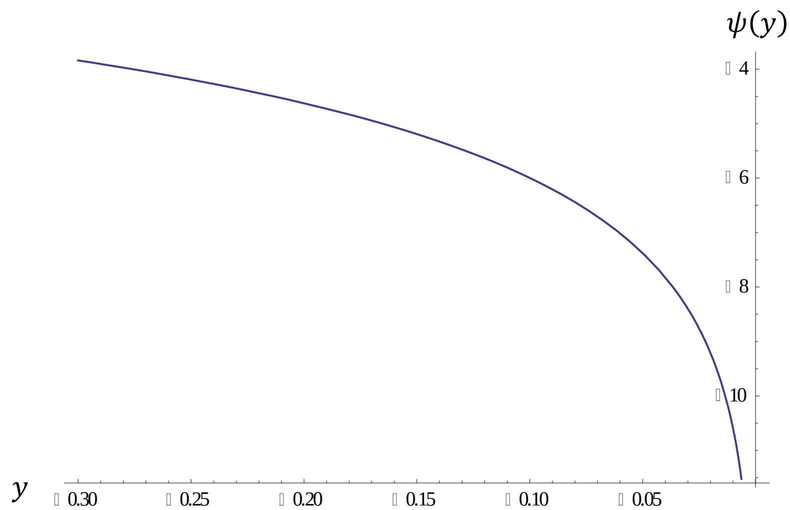



Figure 6.10: Travelling wave solutions of Eq. $(6.56)$ with $z = 0, l = 1$

In Figure 6.10, the wave solution of eq. (6.56) is displayed for various values of the variable $y$, and with $z = 0$ and $l = 1$. When $P_0 = 0$, then from (6.33) the plasma pressure is

$$P(y,z) = \left(-\frac{l\,u_0^2}{8\,\pi} cosh\left[-4\tanh^{-1}\left(e^{\sqrt{l}\,e^{\frac{-z}{l}}\,sin\frac{y}{l}}\right)\right]\right)e^{\frac{-2\,z}{l}}. \qquad (6.57)$$

The plasma pressure solution $(6.57)$ is a special geometric configuration of the previously described systems. An atmospheric pressure plasma jet is a nonthermal, glow discharge plasma functioning at atmospheric pressure and has found growing use in various applications like nanoscience and biodecontamination. The solutions obtained in Fig. 6.11 using the BTs are adequate for describing parallel filaments of diffuse, magnetized plasma pressure. Fig. 6.12 illustrates the deviation of the pressure from the background when $z = 0$, $l = u_0 = 1$.

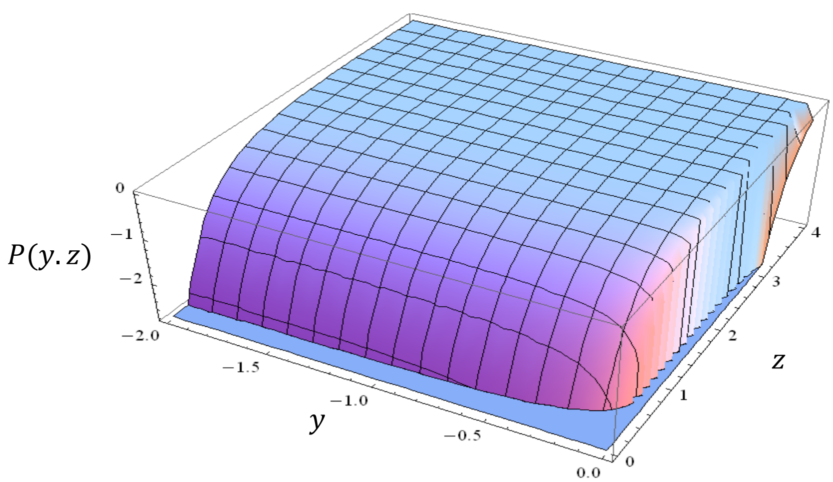



Figure 6.11: The graph of plasma pressure $P(y.z)$

with $\quad l = u_0 = 1$

In Figure 6.11 the Bäcklund transformations procedure has been developed to deduce the plasma pressure and anisotropy from the magnetic field model. Growing interest in microwave atmospheric pressure plasmas calls for efficient and flexible sources of such plasmas.

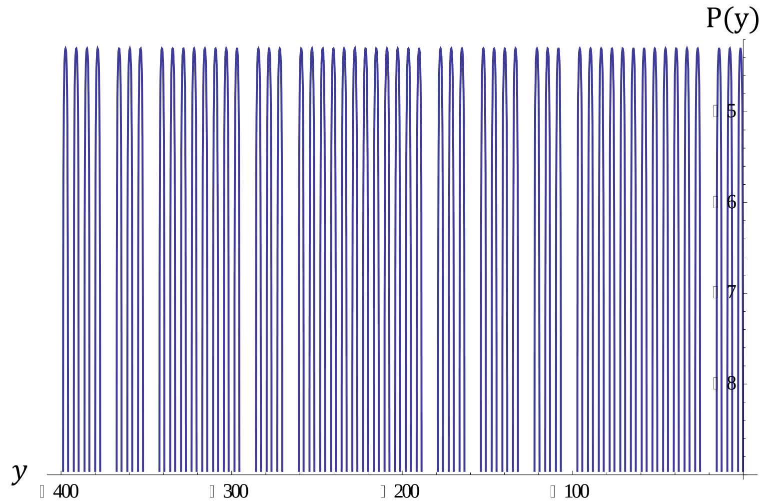



Figure 6.12: The graph of plasma pressure P(y, z)

with $z = 0$, $l = u_0 = 1$

Figure 6.12 shows the profile of plasma pressure (pressure enhancement) as a function of height $y$ for the plane-parallel special case of $z = 0$.

### 6.5 Conclusions

In this chapter, we presented a family of solutions of magnetic fields and plasma distributions in equilibrium in a gravitational field. The usefulness of such simple analytic solutions is that they provide physical insight into more complicated real systems. The main focus of this chapter has been on two classes of nonlinear magnetostatic solutions that are obtained analytically by the sine–cosine and Bäcklund transformations, namely, the solutions corresponding to the particular choice of the pressure profile, given in terms of the magnetic flux function.

This study shows that the magnetic fields influence every aspect of coronal physics. We have presented a new, analytic family of solutions for isothermal magnetostatic atmospheres. Despite the many simplifications of our models, they provide a useful tool for understanding the physics of magnetostatic equilibrium.

# Chapter 7
# Conclusion and further research

## 7.1 Conclusions

In the introductory chapter, a brief description of nonlinear phenomena and a survey on some nonlinear models was presented. We also discussed solitons and various types of travelling wave solutions. A brief account of some important and widely used analytic methods to obtain exact solutions for a variety of nonlinear PDEs relevant to physical problems was also given.

Chapter 3 introduced some important mathematical preliminaries, along with some basic considerations for the models considered in this thesis.

In chapter 4, Sec. 4.2 we introduced our plasma model. In Sec. 4.3, we developed the system of PDEs that described ion acoustic waves in a plasma, and then we derived the KdV equation and obtained exact solutions from a simpler system of PDEs that described ion acoustic waves in a plasma. In Sec. 4.4, we derived the mKdV equation from a simpler system of PDEs that also described ion acoustic waves in a plasma and again obtained exact solutions. Finally, in Sec. 4.5, a summary of the results was presented.

In chapter 5, specifically section 5.1, we reviewed the main governing equations of incompressible MHD. In Section 5.2, the $sin\,(k\xi\,)\ -\ cos\,(k\xi\,)$ method was applied to extract exact solutions for the incompressible MHD problem presented. Finally, Section 5.3 presented a summary of the results obtained in the preceding sections.

In chapter 6, the equations describing the magnetohydrostatic equilibria for a plasma in a gravitational field were investigated analytically. For equilibria with one ignorable spatial coordinate, the equations reduce to a single nonlinear elliptic equation for the magnetic potential known as the Grad–Shafranov equation. Specifying the arbitrary function in the latter equation yielded a nonlinear elliptic equation. Analytical nonlinear periodic solutions of this elliptic equation were obtained for the case of an isothermal atmosphere in a uniform gravitational field which can be useful for modelling the solar atmosphere.

## 7.2 Further research

The present study can further be extended on the following fronts:

- The above-described continuum plasma descriptions and MHD are currently the most widely used plasma models in theoretical investigations, experiments and modelling in astrophysics, thermonuclear fusion research, experimental chemistry, geological sciences, and other areas. However, as can be seen from this review, because of the complexity of these systems, knowledge of their analytical structure is very limited.

- The general existence, uniqueness, and stability of solutions of initial and boundary value problems are unknown, and methods of construction of particular solutions have not been developed. Due to these reasons, most MHD modelling is currently done numerically, usually under dimensional reductions. Fully 3-dimensional dynamic MHD simulations have such a high degree of computational complexity that multiprocessor supercomputers are generally required.

- In numerical simulations, the existence and uniqueness of solutions of interest are usually assumed, which generally might not be the case.

- The stability of soliton solutions is an important issue for practical problems. Thus, stability analysis of the obtained soliton solutions should be considered.
- The problems solved analytically here can be solved numerically as well. Detailed numerical studies can often serve as a check to support the validity of assumptions made in generating analytic solutions.
- One may also be able to generalize the (3+1) dimensional mKdV equation with saturable nonlinearities in order to investigate the dynamical behavior of solitons.

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